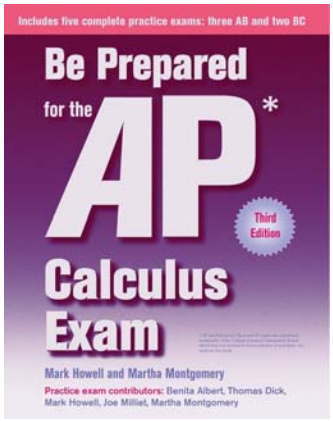

Be Prepared
for the

AP

Calculus Exam

Third Edition

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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The free-response questions for this exam are posted on AP Central, apcentral.collegeboard.org:

- AP Calculus AB
- AP Calculus BC

2025 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

A. $\frac{1}{4}\int_0^4 C(t)\,dt$ ≈ 2.778 acres. [1]

B. $\frac{38}{25+t^2} = \frac{C(4)-C(0)}{4-0} \Rightarrow t \approx 2.154$ weeks. [2]

C. $\lim_{t\to\infty} C'(t) = 0$.

D. $A'(t) = C'(t) - 0.1\cdot\ln(t) = 0 \Rightarrow t = 11.442$. At $t = 11.442$, $A'(t)$ changes sign from positive to negative so there is a local maximum at $t = 11.442$. Since $t = 11.442$ is the only critical point in the interval $4 \le t \le 36$, $A(t)$ attains its absolute maximum at $t = 11.442$. [3]

Notes:

1. No need to mention the units, but they are provided for accuracy.
2. Again, no need to mention the units.
3. This is the "single critical point" argument. The candidates test could also be applied, where we assert that A(t) is continuous, evaluate $A(4), A(11.442)$, and $A(36)$, and select t = 11.442 since $A(11.442)$ is the biggest among these three values. We could also make a global argument. Since $A'(t) > 0$ for $4 < t < 11.442$ and $A'(t) < 0$ for $11.442 < t < 36$, the function A is increasing for $4 < t < 11.442$ and decreasing for $11.442 < t < 36$. Therefore, the maximum occurs at t = 11.442.

Question AB-2

A. $\int_0^3 g(x)-f(x)\,dx$ ▪ ≈ 5.137 📋1

B. $\int_0^3 \big(g(x)-f(x)\big)\cdot x\,dx$ ▪ ≈ 7.705

C. $\pi\int_0^3 \big(g(x)+2\big)^2-\big(f(x)+2\big)^2\,dx$ 📋2

D. $1+\pi\cos(\pi x)=2x-2 \Rightarrow$ ▪ $x=0.676$

📋 **Notes:**

1. Since the functions $f(x)$ and $g(x)$ are needed for all four parts of the problem, store them in the calculator and use the stored functions in all calculations. This is more efficient than retyping the functions' definitions in all four parts, and helps avoid transcription errors.
2. No need to evaluate. For the record, it is approximately 92.226.

Question AB-3

A. $R'(1) \approx \dfrac{R(2)-R(0)}{2-0} = \dfrac{100-90}{2} = 5$ words per minute per minute.

B. Yes. Since R is differentiable, R is also continuous. $R(0) = 90$ and $R(10) = 162$ and $90 < 155 < 162$. The Intermediate Value Theorem [1] guarantees the existance of c, $0 < c < 10$, such that $R(c) = 155$.

C. $2 \cdot \left(\dfrac{90+100}{2} \right) + 6 \cdot \left(\dfrac{100+150}{2} \right) + 2 \cdot \left(\dfrac{150+162}{2} \right)$. [2]

D. $\displaystyle\int_0^{10} W(t)\,dt = \int_0^{10} -\frac{3}{10}t^2 + 8t + 100\,dt = -\frac{t^3}{10} + 4t^2 + 100t \Big|_0^{10} = -\frac{1000}{10} + 400 + 1000 - 0$ [3]

Notes:

1. Naming the theorem is not necessary, but verifying that the conditions required to apply the theorem is.
2. No need to simplify, but for the record, this is 1252.
3. Again, no need to simplify. This is 1300 words.

Question AB-4

A. $g'(8) = f(8) = 1$.

B. $g'(x) = f(x)$ and f has extrema at $x = -3$, $x = 3$, and $x = 6$. At each of these points, the graph of g has a point of inflection. [1]

C. $g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2} \cdot 6 \cdot 3$. [2]

$g(0) = \int_6^{0} f(t)\,dt = -\frac{1}{2} \cdot \pi \cdot 9$. [3]

D. $g'(x) = f(x) < 0$ for $-6 < x < 0$ and $g'(x) \geq 0$ for $0 < x < 12$. Therefore, g decreases for $-6 < x < 0$ and increases for $0 < x < 12$. Thus, g has its absolute minimum at $x = 0$. [4]

Notes:

1. In this setting, points of inflection can be justified in a couple of other ways. One by asserting that $g'(x) = f(x)$ and f changes from increasing to decreasing or vice-versa at each of these points. Another is by stating that $g'(x) = f(x)$, $g''(x) = f'(x)$, and $f'(x)$ changes sign at these points.
2. No need to simplify (the answer is 9).
3. A cleaner looking expression is $-\frac{9\pi}{2}$.
4. Without excluding the existence of other critical points, the argument that $g'(x)$ changes sign from negative to positive at $x = 0$ is not sufficient to justify the absolute minimum. It is a common error on the AP Exams for students to present such an argument.

Question AB-5

A. $v_H(1) = x_h'(1) = (2-4)\cdot e^{-3} = -2e^{-3}$.

B. $v_H(t) = (2t-4)\cdot e^{t^2-4t}$

For $0 < t < 2$, $v_H(t) < 0$ and for $2 < t < 5$, $v_H(t) > 0$.

For $0 < t < 1$, $v_J(t) < 0$ and for $1 < t < 5$, $v_J(t) > 0$. Therefore, the velocities have opposite signs for $1 < t < 2$ and the particles are moving in opposite directions during that time interval. 1

C. Since $v_J(2) > 0$ and $v_J'(2) > 0$, the velocity of particle *J* is positive and increasing. So particle *J* is speeding up at $t = 2$. 2

D. $x_J(2) = x_J(0) + \int_0^2 x_J'(t)\,dt = 7 + \int_0^2 2t(t^2-1)^3\,dt = 7 + \left.\left(\frac{(t^2-1)^4}{4}\right)\right|_0^2 = 7 + \left(\frac{81}{4} - \frac{1}{4}\right)$ 3

Notes:

1. Making parallel sign charts for $v_H(t)$ and $v_J(t)$ on scrap paper helps with the analysis.
2. An assertion that the velocity and acceleration have the same sign is also an acceptable reason for the conclusion that the particle is speeding up.
3. It simplifies to 27.

Question AB-6

A. $3y^2y' - 2yy' - y' + \frac{x}{2} = 0 \Rightarrow y'\left(3y^2 - 2y - 1\right) = -\frac{x}{2} \Rightarrow y' = \frac{dy}{dx} = \frac{-x}{2\left(3y^2 - 2y - 1\right)}$.

B. At $(2,-1)$, $\frac{dy}{dx} = -\frac{2}{2(3+2-1)} = -\frac{1}{4}$. The tangent line is $y + 1 = -\frac{1}{4}(x-2)$.

At $x = 1.6$, $y \approx -\frac{1}{4}(-0.4) - 1$ [1]

C. $3y^2 - 2y - 1 = 0 \Rightarrow (3y+1)(y-1) = 0$. When $y = 1$ or $y = -\frac{1}{3}$, $\frac{dy}{dx}$ is infinite. We have $1^3 - 1^2 - 1 + \frac{1}{4}x^2 = 0 \Rightarrow x = 2$ or $x = -2$. So the point $(2,1)$ is the only point on the curve *G* where the curve has a vertical tangent and $x > 0$ and $y > 0$. $y = 1$ at that point.

D. For curve *H*, $2x\frac{dy}{dt} + 2y\frac{dx}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$. At the point $(4,1)$, we have $8\frac{dy}{dt} + 2 \cdot 1 \cdot 3 + 1\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{6}{9} = -\frac{2}{3}$.

Notes:

1. No need to simplify further. For the record, this is –0.9.

2025 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

A. For curve C, $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} \approx 1.031$. [1]

B. $2\sin^2\theta = \frac{1}{2} \Rightarrow \theta_1 \approx 0.5236$ and $\theta_2 \approx 2.61799$. [2]

The area is $\frac{1}{2}\int_{\theta_1}^{\theta_2}\left(2\sin^2\theta\right)^2 - \left(\frac{1}{2}\right)^2 d\theta \approx 2.067$.

C. $\frac{dx}{d\theta} = 0 \Rightarrow \theta \approx 0.955$. For $0 < \theta < 0.955$, $\frac{dx}{d\theta} > 0$ and for $0.955 < \theta < \frac{\pi}{2}$, $\frac{dx}{d\theta} < 0$. So, at $\theta = 0.955$, the point on the curve is farthest from the y-axis, because x is at its maximum. [3]

D. $\left.\frac{dr}{dt}\right|_{\theta=1.3} = \frac{dr}{d\theta}\cdot\left.\frac{d\theta}{dt}\right|_{\theta=1.3} \approx 1.0310\cdot 15 = 15.465$.

Notes:

1. It is a good idea to enter the functions for the two curves and check that your graph agrees with the graph shown in the problem. Use the stored functions for subsequent calculations. This helps avoid transcription errors.
2. It takes some skill with the calculator to use the names of the functions for the computations. But practicing this is well worth the effort. Here, you could use the names of the polar functions to define functions of X and find intersection points of their graphs in Function mode. Store the limits of integration in calculator variables and use the stored values in your setups.
3. For other justification options, see problem AB1, Part D.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

A. $\dfrac{d^2y}{dx^2} = (3-x)2y\dfrac{dy}{dx} + y^2(-1) = (3-x)2y(3-x)y^2 - y^2$. $f''(1) = (2)(-2)(2)(1) - 1 = -9$.

B. Since $f'(1) = 2$ and $f''(1) = -9$, $P_2(x) = -1 + 2(x-1) - \dfrac{9}{2}(x-1)^2$.

C. A Lagrange Error Bound is $\dfrac{60}{3!}(1.1-1)^3 = \dfrac{10}{1000} = 0.01$. So $\left|P_2(1.1) - f(1.1)\right| \le 0.01$.

D. At $(1,-1)$, $\dfrac{dy}{dx} = 2$. The first step of Euler's method is $y = -1 + 2 \cdot 0.2 = -0.6$. At $(1.2,-0.6)$, $\dfrac{dy}{dx} = 1.8 \cdot (-0.6)^2$. [1] The second step of Euler's method is $y = -0.6 + 1.6 \cdot (-0.6)^2 \cdot 0.2$. [2]

Notes:

1. You don't have to do the arithmetic. But this simplifies to 0.648.
2. More arithmetic that you can leave "as-is." For the record, this simplifies to -0.4704.

Question BC-6

A. Since $\dfrac{a_{n+1}}{a_n}=\dfrac{\dfrac{(x-4)^{n+2}}{(n+2)3^{n+1}}}{\dfrac{(x-4)^{n+1}}{(n+1)3^n}}=\dfrac{(x-4)}{3}\cdot\dfrac{n+1}{n+2}$, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\left|\dfrac{x-4}{3}\right|\lim\limits_{n\to\infty}\dfrac{n+1}{n+2}=\left|\dfrac{x-4}{3}\right|<1\Rightarrow 1<x<7$.

At $x = 7$, the series is $\sum\limits_{n=1}^{\infty}\dfrac{3}{n+1}=3\left(\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\ldots\right)$. This is a constant multiple of the harmonic series minus the first term, and diverges. At $x = 1$, the series is $\sum\limits_{n=1}^{\infty}\dfrac{-3\cdot(-1)^n}{n+1}=-3\left(-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\ldots\right)$. This is a constant multiple of the alternating harmonic series minus the first term, and it converges. The interval of convergence is $1\le x<7$.

B. $f'(x)=\dfrac{x-4}{3}+\dfrac{(x-4)^2}{3^2}+\dfrac{(x-4)^3}{3^3}+\ldots+\dfrac{(x-4)^n}{3^n}+\ldots$

C. The sum is $\dfrac{\dfrac{x-4}{3}}{1-\dfrac{x-4}{3}}=\dfrac{x-4}{3-(x-4)}=\dfrac{x-4}{7-x}$.

D. The radius of convergence for the series for f is 3, so the radius of convergence for f' is also 3. Since the interval of convergence is centered at $x = 4$, $x = 8$ is outside the interval of convergence. So the series diverges at $x = 8$. [1]

Notes:

1. We could also substitute $x = 8$ into the series, producing a geometric series with a common ratio of $\dfrac{4}{3}>1$, so the series diverges.