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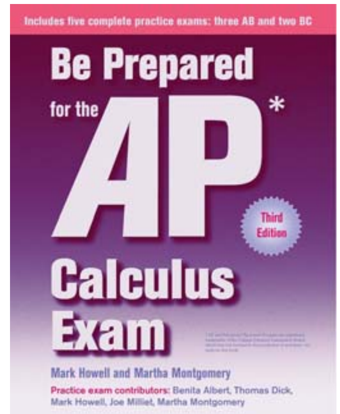
for the

AP

Calculus Exam



Third Edition

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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The free-response questions for this exam are posted on apstudent.collegeboard.org and, for teachers, on AP Central:

- For students: AP Calculus AB
  AP Calculus BC
- For teachers: AP Calculus AB
  AP Calculus BC

2024 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) $C'(5) \approx \dfrac{69-85}{7-3}$ °C/minute. [1]

(b) $\int_0^{12} C(t)\,dt \approx 100 \cdot 3 + 85 \cdot 4 + 69 \cdot 5$. [2] $\dfrac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature in °C of the coffee over the 12 minute time interval.

(c) $C(20) = C(12) + \int_{12}^{20} C'(t)\,dt \approx 55 - 14.6708 \approx 40.329$ °C.

(d) For $12 < t < 20$, $C''(t) > 0$ so the temperature of the coffee is decreasing at an increasing rate. [3]

Notes:

1. No need to simplify, but, for the record, this is –4 °C/minute. Don't forget to mention units.
2. Again, no need to simplify. This is equal to 985. Units are neither requested nor required here.
3. In fact, we know that the temperature is decreasing at an increasing rate, since $C'(t) < 0$ for $12 \le t \le 20$.

Question AB-2

(a) $v(t) = 0 \Rightarrow t_R = 1.426$. For $0 < t < 1.4256$, $v(t) > 0$ so the particle is moving to the right during that time. [1]

(b) The acceleration $a(1.5) = v'(1.5) \approx -1$. Since $a(1.5) < 0$ and $v(1.5) < 0$ the speed of the particle is increasing at $t = 1.5$. [2]

(c) $x(4) = x(1) + \int_1^4 v(t)\,dt \approx -3 + 0.197 = -2.803$.

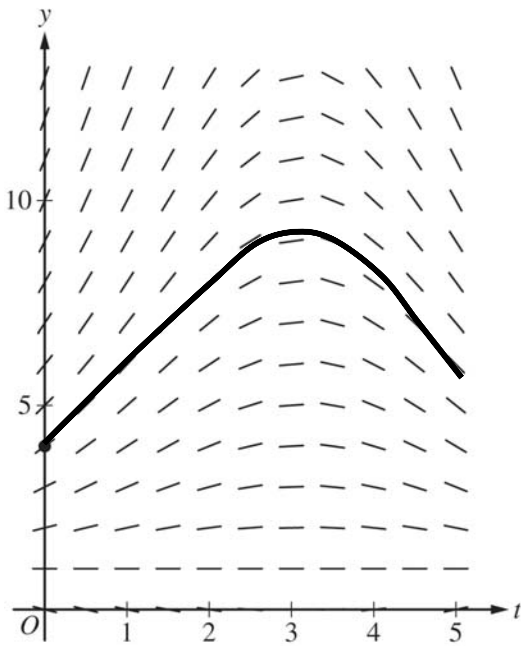
(d) The total distance is $\int_1^4 |v(t)|\,dt \approx 0.958$.

Notes:

1. Since the function $v(t)$ is needed for all four parts of the problem, store the function in your calculator and use the stored function in all the calculations. This is more efficient than retyping the function definition in all four parts, and helps avoid transcription errors.
2. Since this is a calculator active problem, it would be simpler to define a new function s for the speed of the particle, where $s(t) = |v(t)|$. Since $s'(1.5) = 1 > 0$, the speed is increasing at $t = 1.5$.

Question AB-3

(a)



(b) $\frac{dH}{dt} = 0 \Rightarrow \cos\left(\frac{t}{2}\right) = 0$ since $H > 1$. H has a critical number at $t = \pi$. For $0 < t < \pi$, $\frac{dH}{dt} > 0$, and for $\pi < t < 5$, $\frac{dH}{dt} < 0$. Therefore, H has a relative maximum at $t = \pi$.

(c) $\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$ [1]

$\Rightarrow \int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right)dt \quad \Rightarrow \quad \ln(H-1) = \sin\left(\frac{t}{2}\right) + C$. [2] From the initial condition, $\ln(3) = C$, so $\ln(H-1) = \sin\left(\frac{t}{2}\right) + \ln(3) \quad \Rightarrow \quad H(t) = 1 + 3e^{\sin\left(\frac{t}{2}\right)}$.

Notes:

1. Leaving the constant $\frac{1}{2}$ with $\cos\left(\frac{t}{2}\right)$ simplifies the upcoming antidifferentiation.
2. Absolute values are not necessary since we are told that $H > 1$.

Question AB-4

(a) $g(-6)=\int_0^{-6} f(t)\,dt=-12$. $g(4)=\int_0^{4} f(t)\,dt=\frac{1}{2}\cdot 4\cdot 2=4$.

$g(6)=\int_0^{6} f(t)\,dt=4-\frac{1}{2}\cdot 2\cdot 1=3$.

(b) $g'(x)=f(x)=0 \quad\Rightarrow\quad x=4$.

(c) $h(x)=f(x)-f(-6)\Rightarrow h(6)=f(6)-f(-6)=-1-0.5=-1.5$

$h'(x)=f'(x)\Rightarrow h'(6)=f'(6)=-0.5$

$h''(x)=f''(x)\Rightarrow h''(6)=0$ [1]

Notes:

1. Since *f* is linear for $0\le x\le 7$, $f'(6)=-\frac{1}{2}$, the slope of the line. And the second derivative of a linear function is always 0, since the slope of a line is constant.

Question AB-5

(a) At $(2,4)$, $\frac{dy}{dx}=\frac{-2\cdot 2}{3+4\cdot 4}=-\frac{4}{19}$. The tangent line is given by $y=4-\frac{4}{19}\cdot(x-2)$. At $x=3$, $y=4-\frac{4}{19}\cdot(4-3)=\frac{72}{19}$.

(b) A horizontal tangent occurs at a point where $\frac{dy}{dx}=0$. This happens only when $x = 0$. But the point $(0,1)$ is not on the curve because $0^2+3\cdot 1+2\cdot 1^2\neq 48$. So the line $y=1$ is not tangent to the curve at $(0,1)$. [1]

(c) At a vertical tangent, $\frac{dy}{dx}$ is infinite. However, at the point $\left(\sqrt{48},0\right)$, $\frac{dy}{dx}=\frac{-2\sqrt{48}}{3}$, which is not infinite. So the line tangent to the curve at $\left(\sqrt{48},0\right)$ is not vertical.

(d) Differentiating implicitly with respect to t, $3y^2\frac{dy}{dt}+2x\frac{dy}{dx}+2y\frac{dx}{dt}=0$. Substituting $x=4, y=2, \frac{dy}{dt}=-2$, we get $-24-16+4\frac{dx}{dt}=0 \Rightarrow \frac{dx}{dt}=10$. The rate of change of the particle's x-coordinate with respect to time is 10.

Notes:

1. Or, when $y = 1$ on the curve, $x^2+3+2=48 \Rightarrow x=\sqrt{43}$ or $x=-\sqrt{43}$. However, $\frac{dy}{dx}\neq 0$ at either of these points.

Question AB-6

(a) $\int_0^2 \left(f(x)-g(x)\right)dx$. [1]

(b) Each cross section of the solid has area given by $\frac{1}{2}\left(g(x)\right)^2$, so the volume is

$$\frac{1}{2}\int_2^5 \left(x^2-2x\right)^2 dx = \frac{1}{2}\int_2^5 \left(x^4-4x^3+4x^2\right)dx = \frac{1}{2}\left(\frac{x^5}{5}-x^4+\frac{4x^3}{3}\right)\Bigg|_2^5$$

$$=\frac{1}{2}\left[\left(\frac{5^5}{5}-5^4+\frac{500}{3}\right)-\left(\frac{32}{5}-16+\frac{32}{3}\right)\right].$$ [2]

(c) $V=\pi\int_2^5 20^2-\left(20-g(x)\right)^2 dx$.

Notes:

1. Use the names of the functions in the setups instead of transcribing the formulas for the functions.
2. No need to simplify any further. For the record, this is $\frac{1}{2}\left(\frac{500}{3}-\frac{32}{5}+16-\frac{32}{3}\right)=\frac{414}{5}$.

2024 BC
AP Calculus Free-Response Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) The speed at $t = 2$ is $\sqrt{(16-4)^2 + \left(-2+\sqrt{2^{1.2}+20}\right)^2} \approx 12.305$ cm/sec. [1]

(b) The total distance traveled is $\int_0^2 \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt \approx 15.902$ cm. [2]

(c) $y(0) = y(2) + \int_2^0 y'(t)\,dt \approx 6 - 7.174 = -1.174$.

(d) $y'(t) < 0 \Rightarrow 5.222 < t < 8$. [3]

Notes:

1. It is a good idea to enter the functions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into your calculator and use them for subsequent calculations. This helps to avoid transcription errors. It is not necessary to include units, but they are provided here for completeness.
2. It takes some skill with the calculator to use the names of the functions for the computations. But practicing this is well worth the effort. On most calculators, names of parametric functions are global, so they can be used anywhere, including in calculations on the home screen.
3. On the calculator, use the function grapher to find where $-x + \sqrt{x^{1.2} + 10} < 0$.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) $h'(\pi) = \sqrt{1 + \left(f'(\pi)\right)^2} = \sqrt{1+36} = \sqrt{37}$. [1]

(b) $\int_0^{\pi} \sqrt{1 + \left(f'(x)\right)^2}\, dx$ is the length of the curve $y = f(x)$ for $0 \le x \le \pi$.

(c) At $(0,0)$, the slope is 5. So the first step in Euler's method is $0 + 5\pi = 5\pi$. When $x = \pi$, the slope is 6. The second step is $5\pi + 6\pi = 11\pi$. $f(2\pi) \approx 11\pi$.

(d) Use integration by parts:

$u = t + 5 \Rightarrow du = dt;\ dv = \cos\left(\frac{t}{4}\right) \Rightarrow v = 4\sin\left(\frac{t}{4}\right)$.

$$\int (t+5)\cos\left(\frac{t}{4}\right) dt = 4(t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) dt + C = 4(t+5)\sin\left(\frac{t}{4}\right) + 16\cos\left(\frac{t}{4}\right) + C.$$

Notes:

1. Your understanding of the Fundamental Theorem of Calculus is assessed in AP Calculus exams with clocklike regularity.

Question BC-6

(a) At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$. For $n \geq 1$, $\frac{n+1}{n^2} > \frac{1}{n} > 0$. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges. By the direct comparison test, $\sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$ also diverges.

(b) The terms of $\sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ alternate in sign. The ratio of the magnitude of consecutive terms is $\dfrac{\frac{n+2}{(n+1)^2}\left(\frac{1}{2}\right)^{n+1}}{\frac{n+1}{n^2}\left(\frac{1}{2}\right)^n} = \frac{1}{2} \cdot \frac{n^3 + 2n^2}{(n+1)^3} = \frac{1}{2} \cdot \frac{n^3 + 2n^2}{n^3 + 3n^2 + 3n + 1} < 1$, so the terms decrease in magnitude.[1] Since -3 is in the interval of convergence, we know the series converges. The Alternating Series Error bound guarantees that $\left| f(-3) - S_3 \right|$ is less than the first omitted term, which is $\frac{5}{16} \cdot \left(\frac{1}{2}\right)^4 = \frac{5}{256} < \frac{1}{50}$.

(c) $\sum_{n=1}^{\infty} \frac{(n+1)n \cdot x^{n-1}}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)x^{n-1}}{n6^n}$. The radius of convergence is the same as the radius of convergence of the series for *f*, which is 6.

(d) Since $\frac{a_{n+1}}{a_n} = \dfrac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}} = \frac{x^2}{3} \cdot \frac{n^3 + 2n^2}{(n+1)^3}$,

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^2}{3} \right| \lim_{n\to\infty} \frac{n^3 + 2n^2}{(n+1)^3} = \left| \frac{x^2}{3} \right| < 1 \Rightarrow -3 < x^2 < 3$, so the radius of convergence is $\sqrt{3}$.

Notes:

1. We could also show that the terms decrease in magnitude using the derivative:

$$\frac{d}{dx}\left(\frac{x+1}{x^2 2^x}\right) = \frac{x^2 2^x - (x+1)\left(2x2^x + x^2 2^x \ln 2\right)}{\left(x^2 2^x\right)^2} = \frac{-x^2 2^x - 2x2^x - x^3 2^x \ln 2 - x^2 2^x \ln 2}{\left(x^2 2^x\right)^2} < 0.$$