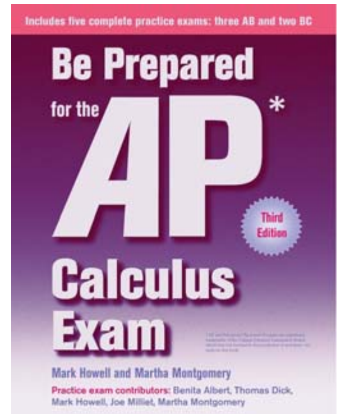

Be Prepared

for the

AP

Calculus Exam

Third Edition

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2023 AB
AP Calculus Free-Response Solutions and Notes

Question AB-1

(a) $\int_{60}^{135} f(t)\,dt$ is the amount of gasoline, in gallons, pumped by the customer from $t = 60$ to $t = 135$ minutes. [1] $\int_{60}^{135} f(t)\,dt \approx 30 \cdot 0.15 + 30 \cdot 0.1 + 15 \cdot 0.05$. [2]

(b) Because f is differentiable, it must also be continuous on [60, 120]. The Mean Value Theorem guarantees the existence of a number c in (60, 120) such that $f'(c) = \dfrac{f(120) - f(60)}{120 - 60} = \dfrac{0.1 - 0.1}{60} = 0$.

(c) $\dfrac{1}{150}\int_{0}^{150} g(t)\,dt \approx 0.096$.

(d) $g'(140) = -0.005$. This means that the rate of flow of gasoline is decreasing at the rate of about 0.005 gallons per second per second at time $t = 140$ seconds. [3]

Notes:

1. Be sure to state the units of the integral (gallons) and the units for t (seconds).
2. No need to simplify, but, for the record, this is 8.25 gallons.
3. Again, be sure to state the units of the derivative (gallons per second per second), as well as the units for t (seconds).

Question AB-2

(a) $v(t) = 0 \Rightarrow t = 56$. Since $v(t)$ changes sign at $t = 56$, Stephen changes direction then. [1]

(b) The acceleration is $a(60) = v'(60) \approx -0.036$ meters per second per second. $v(60) \approx -0.160$. Since $v(60) < 0$ and $v'(60) < 0$, Stephen is speeding up at $t = 60$.

(c) The distance between Stephen's positions at $t = 20$ and $t = 80$ is $\int_{20}^{80} v(t)\,dt \approx 23.384$.

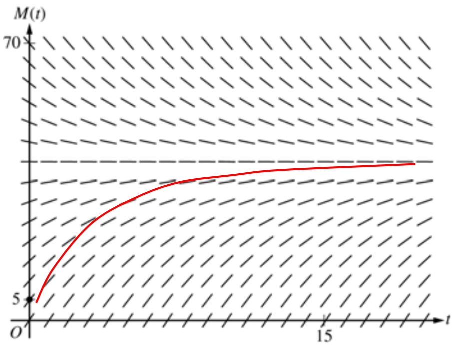
(d) The total distance is $\int_{0}^{90} |v(t)|\,dt \approx 62.164$.

Notes:

1. Since the function $v(t)$ is needed for all four parts of the problem, store the function in the calculator and use the stored function in all the calculations. This is more efficient than retyping the function definition in all four parts, and helps avoid transcription errors.

Question AB-3

(a)



(b) The slope of the tangent line at $t = 0$ is $\frac{1}{4}\cdot(40-5) = \frac{35}{4}$. The tangent line is $y = 5 + \frac{35t}{4}$, so $M(2) \approx 5 + \frac{70}{4}$ degrees centigrade. [1]

(c) $\frac{d^2M}{dt^2} = \frac{d}{dt}\left(\frac{dM}{dt}\right) = \frac{d}{dt}\left(\frac{1}{4}(40-M)\right) = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{16}(40-M)$. On the interval $0 < t < 2$, we know that $M < 40$, and so $\frac{d^2M}{dt^2} < 0$. Therefore, the tangent line gives an overestimate of $M(2)$. [2]

(d) $\frac{dM}{40-M} = \frac{1}{4} \Rightarrow -\ln|40-M| = \frac{t}{4} + C \;\Rightarrow\; -\ln 35 = C$. Substituting, $\ln|40-M| = \ln 35 - \frac{t}{4} \Rightarrow |40-M| = 35e^{-\frac{t}{4}}$. Since $5 \le M(t) < 40$ for $t \ge 0$, the absolute value can be removed, and $M(t) = 40 - 35e^{-\frac{t}{4}}$.

Notes:

1. This simplifies to 22.5 or $\frac{45}{2}$. The units were not requested and are optional.

2. It is necessary to state that $\frac{d^2M}{dt^2} < 0$ on the entire interval, not just at $t = 2$, in order to justify the claim that the tangent line overestimates $M(2)$.

Question AB-4

(a) *f* has neither a relative maximum nor a relative minimum at $t = 6$ because *f* is continuous and differentiable and $f'(x)$ does not change sign at $x = 6$. [1]

(b) The graph of *f* is concave down on $-2 < x < 0$ and $4 < x < 6$ because $f'(x)$ is decreasing on these intervals.

(c) Since *f* is differentiable, it must be continuous, so $\lim_{x\to 2} f(x) = f(2) = 1$.
Thus $\lim_{x\to 2}\left(6f(x) - 3x\right) = 6\cdot 1 - 6 = 0$. Also, $\lim_{x\to 2}\left(x^2 - 5x + 6\right) = 0$. L'Hospital's rule applies, and $\lim_{x\to 2}\dfrac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2}\dfrac{6f'(x)-3}{2x-5} = \dfrac{0-3}{-1} = 3$. [2]

(d) The only minimum points for *f* occur at the left endpoint, $x = -2$, since $f'(x) > 0$ for $-2 \le x < -1$, and at $x = 2$, where $f'(x)$ changes sign from negative to positive. $f(-2) = f(2) + \int_2^{-2} f'(x)\,dx = 1 + (3-1) = 3$, while $f(2) = 1$. Therefore, the absolute minimum value of *f* is 1, reached at $x = 2$.

Notes:

1. Stating that *f* is continuous and differentiable will likely not be required for credit.
2. The following solution is also correct: *f* is differentiable, and $6f(2) - 3\cdot 2 = 0$ and $2^2 - 5\cdot 2 + 6 = 0$. Therefore, using the weaker form of L'Hospital's rule (which applies since the numerator and denominator are both differentiable in a neighborhood of $x = 2$, and the derivative of the denominator is not 0 there), $\lim_{x\to 2}\dfrac{6f(x)-3x}{x^2-5x+6} = \dfrac{6f'(2)-3}{2\cdot 2-5} = \dfrac{0-3}{-1} = 3$.

Question AB-5

(a) $h'(x) = f'(g(x)) \cdot g'(x) \Rightarrow h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$.

(b) $k''(x) = (f(x))^2 \cdot g'(x) + 2g(x)f(x)f'(x) \Rightarrow$
$k''(4) = 4^2 \cdot 2 + 2 \cdot (-3) \cdot 4 \cdot 3 = 32 - 72 = -40$. Since $k''(4) < 0$ and $k''(x)$ is continuous, the graph of k is concave down at $x = 4$. [1]

(c) $m(2) = 40 + \int_0^2 f'(t)\,dt = 40 + f(2) - f(0) = 40 + 7 - 10 = 37$.

(d) $m'(x) = 15x^2 + f'(x) \Rightarrow m'(2) = 60 + f'(2) = 60 - 8 = 52$. So m is increasing at $x = 2$ because $m'(2) > 0$. [2]

Notes:

1. $k''(x)$ is continuous, since all of the functions that define it are continuous, so $k''(x) < 0$ on an interval that contains $x = 4$.
2. Using data about functions presented in a tabular format has become essential in AP Calculus.

Question AB-6

(a) $6x\frac{dy}{dx}+6y=3y^2\frac{dy}{dx} \Rightarrow 6y=\left(3y^2-6x\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{6y}{3y^2-6x}=\frac{2y}{y^2-2x}$.

(b) At a point with the horizontal tangent, $\frac{dy}{dx}=0 \Rightarrow y=0$. On the curve, if $y = 0$, $6x\cdot 0 = 2+0^3 \Rightarrow 0=2$. So there is no point on the curve with a horizontal tangent.

(c) At a vertical tangent, $\frac{dy}{dx}$ is infinite so $y^2=2x \Rightarrow x=\frac{y^2}{2}$ and $y\neq 0$. Substituting into the equation for the curve, $6\cdot\frac{y^2}{2}\cdot y=2+y^3 \Rightarrow 2y^3=2 \Rightarrow y=1$. The point where the curve has a vertical tangent is $\left(\frac{1}{2},1\right)$.

(d) $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$, therefore, at $\left(\frac{1}{2},-2\right)$, $\frac{dy}{dt}=\frac{-4}{(-2)^2-1}\cdot\frac{2}{3}=-\frac{8}{9}$. [1]

Notes:

1. We could treat this as a related rates problem and differentiate implicitly with respect to time: $6x\frac{dy}{dt}+6y\frac{dx}{dt}=3y^2\frac{dy}{dt} \Rightarrow 3\frac{dy}{dt}-12\cdot\left(\frac{2}{3}\right)=12\frac{dy}{dt} \Rightarrow \frac{dy}{dt}=-\frac{8}{9}$.

2023 BC
AP Calculus Free-Response Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) The velocity vector is $\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle e^{\cos t}, 2\cos t \right\rangle$. At $t = 1$, the acceleration vector is

$\vec{a}(1) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle \Big|_{t=1} \approx \langle -1.444, -1.683 \rangle$. [1]

(b) $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5 \Rightarrow t \approx 1.254$. [2]

(c) The slope of the tangent line is $\frac{dy}{dx}\Big|_{t=1} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\Bigg|_{t=1} \approx \frac{1.0808}{1.7165} = 0.630$. [3]

$x(1) = x(0) + \int_0^1 \frac{dx}{dt}\,dt \approx 1 + 2.342 = 3.342$.

(d) The total distance traveled is $\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \approx 6.035$.

Notes:

1. It is a good idea to enter the formulas for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into your calculator and use them for subsequent calculations. This helps avoid transcription errors.
2. It takes some agility with the calculator to define speed as a function, using the previously entered function names from part (a). On most calculators, names of parametric functions are global, and so can be used anywhere. Then find the first positive intersection point between the speed function and $y = 1.5$.
3. Don't do any rounding until you get the final answer.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) The area is $\int_0^3 f(x)-g(x)\,dx = 10-\int_0^3 \frac{12}{3+x}\,dx = 10-\left(12\ln(3+x)\Big|_0^3\right) =$
$10-12\ln 6+12\ln 3$. [1]

(b) $\int_0^\infty \left(\frac{12}{3+x}\right)^2 dx = 144\cdot\lim_{b\to\infty}\int_0^b (3+x)^{-2}\,dx = 144\cdot\lim_{b\to\infty}\left(-\frac{1}{3+x}\right)\Big|_0^b = 144\cdot\left(0+\frac{1}{3}\right) = \frac{144}{3}$. [2]

(c) Integrating by parts, with $u = x, dv = f'(x)\,dx$ we get $du = dx, v = f(x) \Rightarrow$
$\int_0^3 h(x)dx = \left(x\cdot f(x)\right)\Big|_0^3 - \int_0^3 f(x)\,dx = 3\cdot 2-10 = -4$.

Notes:

1. The answer need not be simplified any further, but it is equal to $10-12\ln 2$.
2. It is essential to use proper limit notation in order to earn full credit when evaluating improper integrals.

Question BC-6

(a) $f^{(4)}(x)=\frac{d}{dx}\left(-2x\cdot f'\left(x^2\right)\right)=-2x\cdot f''\left(x^2\right)\cdot 2x-2f'\left(x^2\right)=-4x^2f''\left(x^2\right)-2f'\left(x^2\right)$

$f^{(4)}(0)=-2f'(0)=-6$; $f'''(0)=0$; $f''(0)=-f(0)=-2$; $f'(0)=3$; $f(0)=2$.

The fourth degree Taylor polynomial is $P_4(x)=2+3x-x^2-\frac{6x^4}{4!}$. 1

(b) The Lagrange error bound guarantees that $\left|f(0.1)-P_4(0.1)\right|\le\frac{15\cdot(0.1)^5}{120}=\frac{1}{8}\cdot\frac{1}{10^5}<\frac{1}{10^5}$.

(c) $g'(0)=1\cdot f(0)=2$. $g''(x)=e^xf'(x)+e^xf(x)\Rightarrow g''(0)=f'(0)+f(0)=5$. The second degree Taylor polynomial is $T_2(x)=4+2x+\frac{5}{2}x^2$. 2

Notes:

1. $=2+3x-x^2-\frac{x^4}{4}$.
2. This Taylor polynomial can also be calculated by multiplying the first degree Maclaurin polynomial for e^x by the first two terms of the polynomial for *f* in part (a), and integrating, but this process takes a bit longer to carry out.
   $e^xf(x)=(1+x+...)(2+3x+...)=2+5x+3x^2+...$. So
   $g(x)=g(0)+\int_0^x g'(x)\,dx=g(0)+\int_0^x\left(2+5x+3x^2+...\right)dx=4+2x+\frac{5}{2}x^2+x^3+....$ So
   $T_2(x)=4+2x+\frac{5}{2}x^2$.