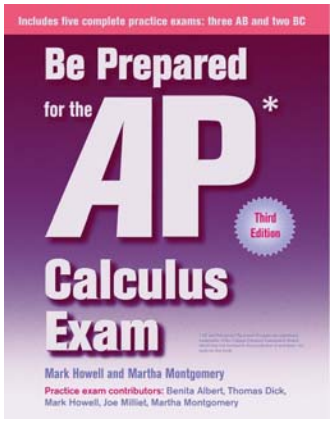

Be Prepared
for the

AP

Calculus Exam

Third Edition

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2022 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) $\int_1^5 A(t)\,dt$. [1]

(b) $\frac{1}{4}\int_1^5 A(t)\,dt \approx 375.537$ vehicles per hour.

(c) $A'(1) \approx 148.947 > 0$, so the rate at which vehicles arrive at the toll plaza is increasing at 6 A.M.

(d) $N'(t) = A(t) - 400 = 0 \Rightarrow t \approx 1.4694$ and $t \approx 3.5977$. A line starts to form at $a = 1.4694$. $N'(t) > 0$ for $1.4694 < t < 3.5977$, and $N'(t) < 0$ for $3.5977 < t < 4$. [2] So the maximum value of $N(t)$ is $N(3.5977) \approx 71.254$. The maximum number of vehicles in line is 71.

Notes:

1. Use the given name for the rate function in your setup, rather than copying its definition. This is faster and avoids transcription errors..
2. We must describe the sign of the derivative over the entire interval from a to 4 in order to justify that the maximum for $N(t)$ occurs at $t = 3.5977$.

Question AB-2

(a) $f(x)=g(x) \Rightarrow$ 🖩 $x=-2, x=0.781975$. Let $a=-2$ and $b=0.781975$. 📋1 The area is $\int_a^b f(x)-g(x)\,dx$ 🖩 ≈ 3.604.

(b) $h'(-0.5)=f'(-0.5)-g'(-0.5)$ 🖩 $=-0.6<0$. So the distance between the curves is decreasing at $x = -0.5$.

(c) The volume is $\int_a^b \left(f(x)-g(x)\right)^2 dx$ 🖩 ≈ 5.340.

(d) The area of a cross-section of the solid is $A=\left(f(x)-g(x)\right)^2$. 📋2

$\left.\frac{dA}{dt}\right|_{x=-0.5} = \left.\frac{dA}{dx}\cdot\frac{dx}{dt}\right|_{x=-0.5}$ 🖩 $\approx -1.32455\cdot 7 \approx -9.272$. 📋3

📋 **Notes:**

1. Store these intersection points into calculator variables and use the stored values in subsequent calculations.
2. Or note that $A(x)=\left(h(x)\right)^2$, so $\frac{dA}{dt}=2h(x)h'(x)\cdot\frac{dx}{dt}$.
3. Note the words, "Find the rate of change of the area …", which tip you off that this is a related rates problem embedded in an area-volume problem.

Question AB-3

(a) $f(0) = f(4) + \int_4^0 f'(t)\,dt = f(4) - \int_0^4 f'(t)\,dt = 3 - \frac{1}{2}(-4\pi) = 3 + 2\pi$.

$f(5) = f(4) + \int_4^5 f'(t)\,dt = 3 + \frac{1}{2} = \frac{7}{2}$.

.

(b) The graph of f has a points of inflection at $x = 2$ and at $x = 6$, because $f'(x)$ has extrema at these points.

(c) $g'(x) = f'(x) - 1$. g is decreasing where $g'(x) < 0$. This occurs on the interval $0 \le x < 5$, so g is decreasing for $0 \le x \le 5$. 1

(d) On the open interval $0 < x < 7$, the only critical point for g occurs at $x = 5$. g has a minimum at this point since $g'(x) = f'(x) - 1$ changes sign from negative to positive there. So $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the minimum value of g on $0 < x < 7$. Since g is continuous, this is also the minimum on the closed interval $0 \le x \le 7$. 2

Notes:

1. Exclusion of endpoints of intervals where a function is increasing or decreasing is generally not penalized, as long as such points are in the domain of the function. So here, a response of $0 \le x < 5$ or $0 < x < 5$ will likely be accepted.
2. The careful analysis here conforms to the language in the Essential Knowledge statement FUN-4.A.8 in the Course and Exam Description.

Question AB-4

(a) $r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{3} = 0.2$ centimeters per day per day. 1

(b) Since r is twice-differentiable, r' is continuous and the Intermediate Value Theorem applies to r' on the interval $0 \le t \le 3$. Since $r'(0) = -6.1$ and $r'(3) = -5.0$, and $-6.1 < -6 < -5.0$, there is a value of t in the interval $0 \le t \le 3$ where $r'(t) = -6$.

(c) $\displaystyle\int_0^{12} r'(t)\,dt \approx (-5 \cdot 3 + (-4.4) \cdot 4 + (-3.8) \cdot 3 + (-3.5) \cdot 2)$. 2

(d) $V = \dfrac{1}{3}\pi r^2 h \;\Rightarrow\; \dfrac{dV}{dt} = \dfrac{1}{3}\pi r^2 \dfrac{dh}{dt} + \dfrac{1}{3} 2\pi r \dfrac{dr}{dt} h$.

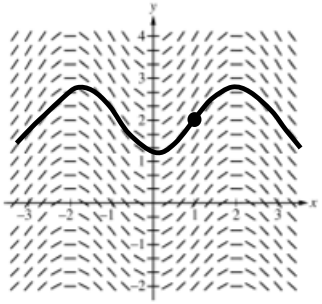
$\left.\dfrac{dV}{dt}\right|_{t=3} = \dfrac{1}{3}\pi \cdot 100^2 (-2) + \dfrac{1}{3} 2\pi \cdot 100 \cdot 50 \cdot (-5)$. 3

Notes:

1. The units of a derivative are always the units of the dependent variable (here, r' with units of centimeters per day) divided by the units of the independent variable (here, t with units of days).
2. For the record, this simplifies to -51, but the simplification is not needed to earn full credit.
3. This simplifies to $-\dfrac{70000\pi}{3}$, but the simplification is not needed to earn full credit.

Question AB-5

(a)



(b) At $(1,2)$ the slope of the tangent line is $\frac{3}{2}$. An equation for the tangent line is $y-2=\frac{3}{2}(x-1)$. At $x=0.8$, $y=2+\frac{3}{2}\cdot(-0.2)$. So $f(0.8)\approx 2+\frac{3}{2}(-0.2)$. [1]

(c) Since $f''(x)>0$ on the interval $[0.8,1]$, the graph of f is concave up on this interval and this approximation is an underestimate for $f(0.8)$.

(d) $\int \frac{dy}{\sqrt{y+7}}=\int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx \Rightarrow 2\sqrt{y+7}=-\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)+C$. From the initial condition, $2\sqrt{2+7}=-\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right)+C \Rightarrow C=6$. [2] Solving for y,

$$\sqrt{y+7}=\left(-\frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)+3\right) \Rightarrow y+7=\left(-\frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)+3\right)^2 \Rightarrow$$

$$y=-7+\left(-\frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)+3\right)^2.$$

Notes:

1. No need to simplify. For the record, this is 1.7.
2. Use the initial condition to find the constant of integration immediately after antidifferentiating to give yourself the best chance to earn credit.

Question AB-6

(a) $v_P(t) = 4e^{-t}$.

(b) $a_Q(t) = \dfrac{-2}{t^3}$. The speed of particle Q is decreasing when its velocity and acceleration have opposite signs. Since $v_Q(t) > 0$ for all t, and $a_Q(t) < 0$ for all $t > 0$, the speed of particle Q is always decreasing when $t > 0$.

(c) $y_Q(t) = 2 + \int_1^t v_Q(u)\,du = 2 + \int_1^t u^{-2}\,du = 2 + \left(-\dfrac{1}{t} + 1\right) = 3 - \dfrac{1}{t}$.

(d) $\lim\limits_{t\to\infty} x_P(t) = 6$ and $\lim\limits_{t\to\infty} y_Q(t) = 3$, so particle P is farther from the origin as $t \to \infty$.

2022 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) $\left.\dfrac{dy}{dx}\right|_{t=4} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=4} = \dfrac{\ln(18)}{\sqrt{17}}$. 1,2

(b) Speed at $t = 4$ is $\sqrt{17 + \left(\ln(18)\right)^2}$. 3 The acceleration vector is

$\vec{a}(4) = \left\langle \dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2} \right\rangle\Big|_{t=4} \approx \langle 0.970, 0.444 \rangle$.

(c) $y(6) = 5 + \int_4^6 \ln\left(2 + t^2\right) dt \approx 11.571$.

(d) Distance traveled = $\int_4^6 \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt \approx 12.136$.

Notes:

1. Easy enough to write the exact slope. The decimal approximation is 0.701, but is not required.
2. It is a good idea to enter the functions for $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ into your calculator and use them for subsequent calculations. This helps avoid transcription errors.
3. Again, it is easy to write the exact speed. Here the decimal approximation is 5.035, but it is not required.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) The area or R is $\int_1^5 \frac{1}{x}\,dx = \ln(5)$.

(b) The volume of the solid is $\int_1^5 xe^{x/5}dx$. Using integration by parts with $u = x, dv = e^{x/5}dx, du = dx, v = 5e^{x/5}$, we have
$\int_1^5 xe^{x/5}dx = 5xe^{x/5}\Big|_1^5 - 5\int_1^5 e^{x/5}dx = \left(5xe^{x/5} - 25e^{x/5}\right)\Big|_1^5 = 25e - 25e - 5e^{1/5} + 25e^{1/5}$. 1

(c) The volume of the solid is
$\int_3^\infty \pi\cdot\left(\frac{1}{x^2}\right)^2 dx = \pi\cdot\lim_{b\to\infty}\int_3^b x^{-4}dx = \pi\cdot\lim_{b\to\infty}\left(-\frac{1}{3x^3}\right)\Big|_3^b = \pi\cdot\left(0+\frac{1}{81}\right) = \frac{\pi}{81}$. 2

Notes:

1. The answer need not be simplified any further, but is equal to $20e^{1/5}$.
2. It is essential to use proper limit notation in order to earn full credit when evaluating improper integrals.

Question BC-6

(a) The ratio test gives $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right| = x^2 \cdot \frac{2n+1}{2n+3} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2 < 1$ when $-1 < x < 1$.

At $x = -1$ the series is $\sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1}$; at $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. The terms of each alternate in sign and decrease in magnitude to 0. Therefore, each of the series converges by the alternating series test, and the interval of convergence is $-1 \le x \le 1$.

(b) $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right|$ is the error from using the first term of the series with $x = \frac{1}{2}$ to approximate $f\left(\frac{1}{2}\right)$. Since the series $\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \ldots$ converges by the alternating series test (the terms alternate in sign and decrease in magnitude to 0), the error does not exceed the absolute value of the first omitted term, which is $\frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{24} < \frac{1}{10}$. So $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$.

(c) $f'(x) = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots$.

(d) $f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \ldots + (-1)^n \left(\frac{1}{6}\right)^{2n} + \ldots$. This is a geometric series with the first term 1 and the common ratio $-\frac{1}{36}$. Therefore, $f'\left(\frac{1}{6}\right) = \frac{1}{1 + \frac{1}{36}} = \frac{36}{37}$.