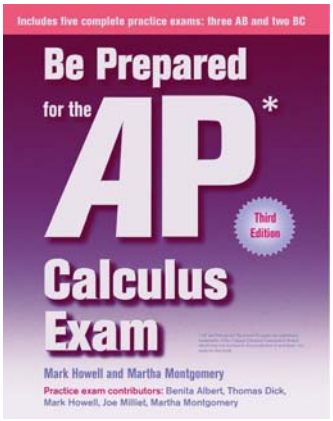

Be Prepared
for the

AP

Calculus Exam

Third Edition

* AP and Advanced Placement Program are registered trademarks of the College Entrance Examination Board, which was not involved in the production of and does not endorse this book.

Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

Chapter 10. Annotated Solutions to Past Free-Response Questions

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

2019 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) $\int_0^5 E(t)\,dt$ ▪ ≈ 153.4577. [1] 153 fish entered the lake from midnight to 5 A.M.

(b) $\frac{1}{5}\int_0^5 L(t)\,dt$ ▪ ≈ 6.059 fish per hour. [1]

(c) The number of fish in the lake is given by $F(t) = F_0 + \int_0^t \big(E(u) - L(u)\big)\,du$, where F_0 is the number of fish in the lake a midnight. Solving $F'(t) = E(t) - L(t) = 0$ ▪ gives $t \approx 6.20356$ hours. Let $b = 6.20356$. [2] The function $F(t)$ is continuous on $0 \le t \le 8$, so, by the Extreme Value Theorem, $F(t)$ must have an absolute maximum on $0 \le t \le 8$, and this maximum can only occur at an endpoint or a critical point. [3] $F(0) = F_0$, $F(b)$ ▪ $\approx F_0 + 135.0149$, and $F(8)$ ▪ $\approx F_0 + 80.920$. The greatest number of fish in the lake occurs at $t = 6.204$ hours after midnight.

(d) The rate of change of the number of fish in the lake is $F'(t) = E(t) - L(t)$, and this rate of change is changing at the rate of $F''(t) = E'(t) - L'(t)$. At 5 A.M., $F''(5)$ ▪ $\approx -10.723 < 0$, so the rate of change is decreasing.

Notes:

1. Use the given name for the rate function in your setup, rather than copying its definition. This is faster and avoids transcription errors. Also, define the functions in your calculator, and use the calculator's names for subsequent computations.
2. Store this intermediate result in a calculator variable and use that variable when evaluating the integral later.
3. This level of detail in a justification of an absolute extremum has not been required on past AP exams. However, the recent emphasis on verifying that the hypothesis of a theorem is satisfied would seem to require here the assertion that the function *F* is continuous. Otherwise, the Extreme Value Theorem would not apply and the candidate test could not be used.

Question AB-2

(a) $v_P(t)$ is differentiable and therefore continuous on $[0.3, 2.8]$. Since $\frac{v_P(2.8)-v_P(0.3)}{2.8-0.3}=\frac{55-55}{2.5}=0$, the Mean Value Theorem guarantees that there is at least one t in the open interval $(0.3, 2.8)$ such that $v_P'(t)=0$. 1

(b) The trapezoidal sum is $\left(\frac{55+0}{2}\right)\cdot 0.3+\left(\frac{-29+55}{2}\right)\cdot 1.4+\left(\frac{55-29}{2}\right)\cdot 1.1=40.75$. 2

(c) $v_Q(t)\ge 60 \Rightarrow$ calc $t\in[1.8662, 3.5192]$. Let $a = 1.8662$ and $b = 3.5192$. 3 The distance traveled by particle Q on this time interval is $\int_a^b v_Q(t)\,dt$ calc ≈ 106.109 meters.

(d) At $t = 2.8$, particle Q has position $-90+\int_0^{2.8} v_Q(t)\,dt \approx 45.9377$. Since particle P started at the origin, its approximate position at $t = 2.8$ is given by the result of Part (b), $\left(\frac{55+0}{2}\right)\cdot 0.3+\left(\frac{-29+55}{2}\right)\cdot 1.4+\left(\frac{55-29}{2}\right)\cdot 1.1=40.75$. So, the particles are $45.9377-40.75$ meters apart. 4

Notes:

1. Recent emphasis on asserting that the hypothesis of a theorem is satisfied when a theorem is applied in a justification requires the mention of the continuity of $v_p(t)$.
2. In general, there is no need to evaluate such expressions, but here the number will be used in Part (d).
3. Store these numbers in calculator variables using the full precision of the calculator.
4. We could just copy the formula from Part (b) here instead of 40.75, but then we would need to take the absolute value of the difference, to be on the safe side. Either way, there is no need to calculate the difference. The distance is about 5.188.

Question AB-3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + \left(2 + \left(9 - \frac{9\pi}{4}\right)\right) \Rightarrow$
$\int_{-6}^{-2} f(x)\,dx = \frac{9\pi}{4} - 4.$

(b) $\int_{3}^{5} \left(2f'(x) + 4\right)dx = 2f(x) + 4x\Big|_{3}^{5} = \left(2f(5) + 20\right) - \left(2f(3) + 12\right) =$
$8 - 2f(3) = 2 + 2\sqrt{5}.$

(c) Since g is continuous, the Extreme Value Theorem applies to g on $[-2, 5]$. Since f is continuous, the Fundamental Theorem applies to g and $g'(x) = f(x)$. 1 The absolute maximum value of g on the interval must occur at an endpoint or an interior critical point.

x	$g(x)$
–2	0
–1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

The absolute maximum value of g is $11 - \frac{9\pi}{4}$.

(d) The function $\frac{10^x - 3f'(x)}{f(x) - \arctan(x)}$ is continuous at $x = 1$, so
$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan(x)} = \frac{10 - 3 \cdot 2}{1 - \arctan(1)}$. 2

Notes:

1. Here again, this level of rigor has not been required in the past on problems like this. However, both the Fundamental Theorem and the Extreme Value Theorem are needed for this justification, so it is prudent to assert that the hypothesis of each is true.

2. You can leave it at that. $\arctan(1) = \frac{\pi}{4}$, of course. It is not equal to 1, so the denominator of the fraction is not 0.

Question AB-4

(a) $V = \pi h \Rightarrow \frac{dV}{dt} = \pi \frac{dh}{dt} = -\frac{\pi}{10}\sqrt{h}$. At h =4, $\frac{dV}{dt} = -\frac{\pi}{5}$ cubic feet per second.

(b) $\frac{d^2h}{dt^2} = -\frac{1}{10} \cdot \left(\frac{1}{2\sqrt{h}}\right) \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$. Since $\frac{d^2h}{dt^2} > 0$ for all t, the rate of change of the height of the water is increasing when the height is 3 feet.

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10}dt \Rightarrow \int h^{-1/2}dh = -\frac{1}{10}\int dt \Rightarrow 2\sqrt{h} = -\frac{t}{10} + C$. Since $h(0) = 5, C = 2\sqrt{5}$ and $\sqrt{h} = \sqrt{5} - \frac{t}{20}$. So $h(t) = \left(\sqrt{5} - \frac{t}{20}\right)^2$. 1

Notes:

1. The domain of h is $0 \le t \le 20\sqrt{5}$, since the model doesn't apply once the tank is empty.

Question AB-5

(a) Area $= \int_0^2 \left(h(x) - g(x)\right) dx = \int_0^2 \left(6 - 2(x-1)^2 - \left(3\cos\left(\frac{\pi x}{2}\right) - 2\right)\right) dx$

$$= \left(6x - 2\frac{(x-1)^3}{3} - \left(\frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right) - 2x\right)\right)\Bigg|_0^2$$

$$= \left(8x - \frac{2}{3}(x-1)^3 - \frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Bigg|_0^2 = \left(16 - \frac{2}{3} - 0\right) - \left(\frac{2}{3}\right).$$ [1]

(b) Volume $= \int_0^2 \frac{1}{x+3} dx = \ln(x+3)\Big|_0^2 = \ln(5) - \ln(3)$.

(c) Each cross section is a "washer" with outer radius $6 - g(x)$ and inner radius $6 - h(x)$. The volume is $\pi\int_0^2 \left(\left(6 - g(x)\right)^2 - \left(6 - h(x)\right)^2\right) dx$.

Notes:

1. This simplifies to $\frac{44}{3}$, but the simplification is not required for full credit.

Question AB-6

(a) $h'(2) = \frac{2}{3}$, the slope of the given tangent line at $x = 2$.

(b) $a'(x) = 3x^3 \cdot h'(x) + 9x^2 \cdot h(x) \;\Rightarrow\; a'(2) = 24 \cdot \frac{2}{3} + 36 \cdot 4$. [1]

(c) Since $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule and $x^2 - 4 = 0$ when $x = 2$, it must be true that $\lim_{x \to 2} \left(1 - \left(f(x)\right)^3\right) = 0$. Since f is differentiable, f must be continuous, and $\left(1 - \left(f(x)\right)^3\right)$ is also continuous. So, $\left(1 - \left(f(2)\right)^3\right) = 0$ and $f(2) = 1$. Since h is differentiable, it is continuous, and $\lim_{x \to 2} h(x) = h(2) = 4$.

Applying L'Hospital's Rule, we get $\lim_{x \to 2} \frac{x^2 - 4}{1 - \left(f(x)\right)^3} = \lim_{x \to 2} \frac{2x}{-3\left(f(x)\right)^2 f'(x)}$. Since f' is differentiable, it is also continuous, and $\lim_{x \to 2} \frac{2x}{-3\left(f(x)\right)^2 f'(x)} = \frac{4}{-3 \cdot 1^2 \cdot f'(2)} = 4$. Therefore, $f'(2) = -\frac{1}{3}$.

(d) We know that $g(2) \le k(2) \le h(2)$, so $4 \le k(2) \le 4$ and $k(2) = 4$. Functions g and h are both continuous, so $\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = 4$. By the Squeeze Theorem, $\lim_{x \to 2} k(x) = 4$. Since $\lim_{x \to 2} k(x) = k(2)$, k is continuous at $x = 2$.

Notes:

1. This simplifies 160, but the simplification is not required for full credit.

2019 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See as AB Question 1.

Question BC-2

(a) Area $= \frac{1}{2}\int_0^{\sqrt{\pi}} \left(r(\theta)\right)^2 d\theta = \frac{1}{2}\int_0^{\sqrt{\pi}} 9\theta \sin^2\left(\theta^2\right) d\theta \approx 3.534$. [1]

(b) The average distance is $\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} r(\theta)\, d\theta \approx 1.580$.

(c) $\frac{1}{2}\int_0^{\arctan(m)} \left(r(\theta)\right)^2 d\theta = \frac{1}{4}\int_0^{\sqrt{\pi}} \left(r(\theta)\right)^2 d\theta$. [2]

(d) $\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} \left(r(\theta)\right)^2 d\theta \approx 3.324$.

Notes:

1. Since you'll need it in several parts of the problem, enter the function $r(\theta)$ into your calculator, and use the stored function name for the calculations.

2. Or $\frac{1}{2}\int_0^{\arctan(m)} \left(r(\theta)\right)^2 d\theta = \frac{1}{2}\int_{\arctan(m)}^{\sqrt{\pi}} \left(r(\theta)\right)^2 d\theta$.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) $f'(x) = -\dfrac{2x-2}{\left(x^2-2x+k\right)^2}$. Since $f'(0) = 6$, $\dfrac{2}{k^2} = 6 \Rightarrow k = \dfrac{1}{\sqrt{3}}$.

(b) $\displaystyle\int_0^1 \frac{1}{x^2-2x-8}\,dx = \int_0^1 \frac{1}{(x-4)(x+2)}\,dx$. We use partial fractions to find an antiderivative:

$\dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 1 = A\cdot(x+2) + B\cdot(x-4)$. Let $x = 4 \Rightarrow A = \dfrac{1}{6}$. Let $x = -2 \Rightarrow B = -\dfrac{1}{6}$. So, $\displaystyle\int_0^1 \frac{1}{x^2-2x-8}\,dx = \int_0^1 \frac{1}{(x-4)(x+2)}\,dx =$

$\dfrac{1}{6}\displaystyle\int_0^1 \left(\frac{1}{x-4} - \frac{1}{x+2}\right)dx = \frac{1}{6}\Big(\ln|x-4| - \ln|x+2|\Big)\Big|_0^1 = -\frac{\ln(2)}{6}$.

(c) $\displaystyle\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \lim_{a\to 1^-}\int_0^a (x-1)^{-2}\,dx + \lim_{b\to 1^+}\int_b^2 (x-1)^{-2}\,dx$

$= \displaystyle\lim_{a\to 1^-}\left(\frac{-1}{x-1}\Big|_0^a\right) + \lim_{b\to 1^+}\left(\frac{-1}{x-1}\Big|_b^2\right) = \lim_{a\to 1^-}\left(\frac{-1}{a-1} - 1\right) + \lim_{b\to 1^+}\left(-1 + \frac{1}{b-1}\right)$. The limits do not exist so the integral diverges.

Question BC-6

(a) From the tangent line, $f(0)=3$ and $f'(0)=-2$. The third degree Taylor polynomial for f at $x=0$ is $3-2x+\frac{3}{2}x^2-\frac{23}{12}x^3$.

(b) The first three terms of the Maclaurin series for e^x are $1+x+\frac{1}{2}x^2$. The Taylor series for $e^x f(x)$ is $\left(1+x+\frac{1}{2}x^2+\ldots\right)\left(3-2x+\frac{3}{2}x^2+\ldots\right)=3-2x+3x+\frac{3}{2}x^2-2x^2+\frac{3}{2}x^2+\ldots$
$=3+x+x^2+\ldots$, so the second degree Taylor polynomial for $e^x f(x)$ is $3+x+x^2$.

(c) $h(x)\approx\int_0^x\left(3-2t+\frac{3}{2}t^2-\frac{23}{12}t^3\right)dt=3x-x^2+\frac{1}{2}x^3-\frac{23}{48}x^4$, so $h(1)\approx 3-1+\frac{1}{2}-\frac{23}{48}$. [1]

(d) $h(1)\approx 3-1+\frac{1}{2}-\frac{23}{48}$. The first omitted term in the series for *h* is $\int_0^1\frac{54}{4!}\cdot t^4dt=\frac{54}{24}\cdot\frac{1^5}{5}=\frac{9}{20}=0.45$, and this is a bound on the error.

Notes:

1. This is $\frac{97}{48}$, but the given answer is sufficient for full credit.