

Be Prepared
for the

AP

Calculus
Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2013 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

- (a) $G'(5) \approx -24.588$ tons per hour per hour.^{□1} At $t = 5$, the rate at which gravel arrives at the processing plant is decreasing at the rate of 24.588 tons per hour per hour.^{□1}
- (b) $\int_0^8 G(t) dt \approx 825.551$ tons of gravel.
- (c) The rate of change of the amount of unprocessed gravel is determined by the rate of change at which gravel arrives, $G(t)$, minus the rate of change at which gravel is processed, which is 100. $G(5) - 100 \approx 98.141 - 100 = -1.859 < 0$. Since this value is negative, the amount of unprocessed gravel is decreasing at $t = 5$ hours.
- (d) The function $A(t) = 500 + \int_0^t (G(x) - 100) dx$ represents the amount of gravel at the plant at time t , for $0 \leq t \leq 8$. $A'(t) = G(t) - 100 = 0$ only at $t = 4.923$.^{□2} Using the candidate test^{□3}, we have $A(0) = 500$, $A(8) = 500 + 825.551 - 800 = 525.551$, and $A(4.923) = 500 + \int_0^{4.923} (G(x) - 100) dx \approx 635.376$. The maximum amount of gravel is $A(4.923) \approx 635.376$ tons.

Notes:

1. It is important to mention the units both for t (hours) and $G(t)$ (tons per hour); $G'(t)$ is measured in tons per hour per hour.
2. Save this value in a calculator variable for later use in evaluating the integral.

Continued ↗

3. In general, using the candidate test is safer than justifying an absolute extremum using the signs of the derivative. Two alternative justifications would earn full credit here: (1) that $A'(t) > 0$ for $0 < t < 4.923$ and $A'(t) < 0$ for $4.923 < t < 8$; or (2) that the only sign change for $A'(t)$ occurs at $t = 4.923$ and $A'(t)$ changes sign from positive to negative there. The argument that $A'(t)$ changes sign from positive to negative at $t = 4.923$ would be, by itself, insufficient.
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Question AB-2

- (a) Solving $|v(t)| = 2$ for $2 \leq t \leq 4$ gives $t \approx 3.128$ and $t \approx 3.473$.
- (b) $s(t) = 10 + \int_0^t v(x) dx$. $s(5) \approx 10 + \int_0^5 v(x) dx \approx -9.207$
- (c) $s'(t) = v(t)$ changes sign at $t \approx 0.536$ and $t \approx 3.318$. The particle changes direction at these times.
- (d) $v(4) \approx -11.476$ and $v'(4) \approx -22.296$. Since both of these values are negative, the speed is increasing at $t = 4$. \square^1

 \square Notes:

1. Since $v(4)$ is negative and decreasing, the magnitude of $v(t)$, that is, speed, is increasing.
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Question AB-3

(a) $C'(3.5) \approx \frac{12.8-11.2}{4-3}$ ☐₁ ounces/minute.

(b) Since C is differentiable, the Mean Value Theorem guarantees the existence of a value t in $(2, 4)$ at which $C'(t) = \frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = 2$.

(c) $\frac{1}{6} \cdot 2 \cdot (5.3+11.2+13.8)$. ☐₂ $\frac{1}{6} \int_0^6 C(t) dt$ is the average amount of coffee in the cup, in ounces, between time $t = 0$ and $t = 6$ minutes.

(d) $B'(t) = (-16)(-0.4)e^{-0.4t}$. So $B'(5) = (-16)(-0.4)e^{-0.4 \cdot 5}$. ☐₃

☐ Notes:

1. $= 1.6$

2. Leave it at that. (It is equal to 10.1.)

3. No need to simplify. If you insist, it is $= \frac{6.4}{e^2}$.

Question AB-4

- (a) f has a minimum at $x = 6$ because $f'(x)$ changes sign from negative to positive there.
- (b) f has a local minimum at $x = 0$ since $f'(x) > 0$ to the right of 0, and another one at $x = 6$. $f(0) = f(8) + \int_8^0 f'(t) dt = f(8) - \int_0^8 f'(t) dt = 4 - (2 + 6 - 3 + 7) = -8$ and $f(6) = f(8) - \int_6^8 f'(t) dt = 4 - 7 = -3$. The absolute minimum value of f is -8 .
- (c) f is concave down and increasing where $f'(x)$ is decreasing and positive. This occurs on the intervals $0 < x < 1$ and $3 < x < 4$.
- (d) $g'(x) = 3(f(x))^2 f'(x)$. $g'(3) = 3(f(3))^2 f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4$. \square_1

Notes:

1. $= 75$

Question AB-5

- (a) Area $= \int_0^2 \left(4 \cos\left(\frac{1}{4}\pi x\right) - (2x^2 - 6x + 4) \right) dx = \int_0^2 \left(4 \cos\left(\frac{1}{4}\pi x\right) - 2x^2 + 6x - 4 \right) dx = \left(\frac{16}{\pi} \sin\left(\frac{\pi x}{4}\right) - \frac{2}{3}x^3 + 3x^2 - 4x \right) \Big|_0^2 = \frac{16}{\pi} - \frac{16}{3} + 12 - 8$. \square_1
- (b) Volume $= \pi \int_0^2 \left((4 - f(x))^2 - (4 - g(x))^2 \right) dx$.
- (c) Volume $= \int_0^2 (g(x) - f(x))^2 dx$.

Notes:

1. $= \frac{16}{\pi} - \frac{4}{3}$

Question AB-6

- (a) At the point $(1, 0)$, $\frac{dy}{dx} = e^0(3 \cdot 1^2 - 6 \cdot 1) = -3$. An equation of the tangent line is $y = -3(x - 1)$. At $x = 1.2$, $y = -0.6$, so $f(1.2) \approx -0.6$. \square^1
- (b) Separating variables, we get $e^{-y} dy = (3x^2 - 6x) dx$. Therefore,
 $\int e^{-y} dy = \int (3x^2 - 6x) dx \Rightarrow -e^{-y} = x^3 - 3x^2 + C$. Substituting the initial condition $y(1) = 0$ we get $-1 = 1 - 3 + C \Rightarrow C = 1 \Rightarrow$
 $-e^{-y} = x^3 - 3x^2 + 1 \Rightarrow e^{-y} = -x^3 + 3x^2 - 1 \Rightarrow y = -\ln(-x^3 + 3x^2 - 1)$. \square^2

 \square **Notes:**

1. The statement $f(1.2) = -0.6$ is incorrect and usually penalized.
 2. The domain of this particular solution must be restricted to the largest interval where $-x^3 + 3x^2 - 1 > 0$ that contains $x = 1$, approximately $(0.653, 2.879)$.
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2013 BC
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Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) $\text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (4 - 2 \sin(\theta))^2 d\theta + \frac{1}{2} \left(\pi + \frac{\pi}{6} + \frac{\pi}{6} \right) \cdot 3^2 \approx 24.709.$

(b) $r \cos \theta = -1 \Rightarrow (4 - 2 \sin(t^2)) \cdot \cos(t^2) = -1 \Rightarrow t \approx 1.428.$

(c) The position vector is $\vec{s}(t) = \left((4 - 2 \sin(t^2)) \cdot \cos(t^2), (4 - 2 \sin(t^2)) \cdot \sin(t^2) \right).$

The velocity vector is

$$\vec{v}(t) = \vec{s}'(t) = \left(\left((4 - 2 \sin(t^2)) \cdot \cos(t^2) \right)', \left((4 - 2 \sin(t^2)) \cdot \sin(t^2) \right)' \right) \Rightarrow \vec{v}(1.5) \approx (-8.072, -1.673).$$

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) Since $\lim_{x \rightarrow 0} (f(x) + 1) = 0$ and $\lim_{x \rightarrow 0} \sin(x) = 0$, by l'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{f(x) + 1}{\sin(x)} = \lim_{x \rightarrow 0} \left(\frac{dy/dx}{\cos(x)} \right) = \frac{y^2(2x+2)|_{x=0, y=-1}}{\cos(0)} = \frac{1 \cdot 2}{1} = 2.$$

(b) At $(0, -1)$, $\frac{dy}{dx} = 2$. The first step gives $y = -1 + 2 \cdot \frac{1}{4} = -\frac{1}{2}$. At $\left(\frac{1}{4}, -\frac{1}{2}\right)$,

$$\frac{dy}{dx} = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}. \text{ The second step gives } y = -\frac{1}{2} + \frac{5}{8} \cdot \frac{1}{4} = -\frac{16}{32} + \frac{5}{32} = -\frac{11}{32}. \text{ So}$$

$$f\left(\frac{1}{2}\right) \approx -\frac{11}{32}. \quad \square_1$$

(c) Separating variables, \square_2 we get

$$y^{-2} dy = (2x+2) dx \Rightarrow \int y^{-2} dy = \int (2x+2) dx \Rightarrow -y^{-1} = x^2 + 2x + C.$$

Substituting the initial condition,

$$1 = 0 + C \Rightarrow C = 1 \Rightarrow -y^{-1} = x^2 + 2x + 1 \Rightarrow y = -\frac{1}{x^2 + 2x + 1}. \text{ The solution is}$$

$$f(x) = -\frac{1}{x^2 + 2x + 1}. \quad \square_3$$

Notes:

1. The statement $f\left(\frac{1}{2}\right) = -\frac{11}{32}$ (an exact equality) is incorrect. Using the exact equality in such situations is usually penalized.
 2. Strictly speaking, we must check first that the solution $y = 0$ does not satisfy the initial condition, but this is usually not the case in AP exam questions and is not tested.
 3. The domain is $x > -1$. It is the largest interval where the function is defined and that contains $x = 0$.
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Question BC-6

(a) $P_1(x) = f(0) + f'(0) \cdot x = -4 + f'(0) \cdot x$. So $-3 = -4 + f'(0) \cdot \frac{1}{2} \Rightarrow f'(0) = 2$.

(b) $P_3(x) = -4 + 2x - \frac{\frac{2}{3}x^2}{2!} + \frac{\frac{1}{3}x^3}{3!} = -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$.

(c) The third degree Taylor Polynomial for $h'(x) = f(2x)$ is

$$P_3(2x) = -4 + 4x - \frac{4}{3}x^2 + \frac{8}{18}x^3, \text{ so the third degree Taylor Polynomial for } h(x) \text{ is}$$

$$Q_3(x) = -4x + 2x^2 - \frac{4}{9}x^3 + C. \text{ Since } h(0) = 7,$$

$$C = 7 \Rightarrow Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3.$$
