Be Prepared
for the

AP

Calculus Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2010 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1 [1]

(a) $\int_0^6 f(t)\,dt \approx 142.275$.

(b) $f(8) - g(8) \approx 48.417 - 108 = -59.583 \text{ ft}^3/\text{hr}$.

(c) $h(t) = \begin{cases} 0, \text{ for } 0 \le t < 6 \\ 125(t-6), \text{ for } 6 \le t < 7 \\ 125 + 108(t-7), \text{ for } 7 \le t \le 9 \end{cases}$

(d) $\int_0^9 f(t)\,dt - h(9) \approx 367.335 - (125 + 2 \cdot 108) = 26.335$.

Notes:

1. Students should expect that Part A of the free-response section of the exam (the calculator part) will continue to involve questions with a real-world context.

Question AB-2

(a) The rate at which entries were being deposited (in hundreds of entries per hour) is, approximately, $\dfrac{E(7)-E(5)}{7-5}=\dfrac{21-13}{2}=\dfrac{8}{2}=4$.

(b) The approximation is $\dfrac{1}{8}\left(2\cdot\dfrac{0+4}{2}+3\cdot\dfrac{4+13}{2}+2\cdot\dfrac{13+21}{2}+1\cdot\dfrac{21+23}{2}\right)$.[1] This represents the average number of <u>hundreds</u> of entries in the box over the 8 hours.

(c) $E(8)-\int_8^{12} P(t)\,dt \approx 23-16.000=7$ hundred entries**.**

(d) The question asks for the maximum value of $P(t)$. $P'(t)=0$ at t = 9.1835 and t = 10.816. The maximum value of $P(t)$ must occur at one of these, or at the endpoints, t = 8 or t = 12.[2] $P(9.1835)$ = 5.089, $P(10.816)$ = 2.911, $P(8)$ = 0, and $P(12)$ = 8. The maximum is at t = 12.

Notes:

1. No need to evaluate. For the curious, the value is $\dfrac{1}{8}(85.5)=10.6875$.
2. The candidate test is the simplest way to find the maximum.

Question AB-3

(a) The number of people who arrived at the ride between $t = 0$ and $t = 3$ is
$\int_0^3 r(t)\,dt = \frac{1000+1200}{2}\cdot 2 + \frac{1200+800}{2}\cdot 1 = 3200$.

(b) For $2 \le t < 3$ people arrive at the rate greater than 800 people/hour and are removed from the line at the constant rate of 800 people/hour. The rate of arrival is greater than the rate of removal, so the number of people in the line is increasing.

(c) Let $P(t)$ be the number of people in the line at time t. Then $P'(t) = r(t) - 800$.
$P'(t) > 0$ for $0 < t < 3$, and $P'(t) < 0$ for $3 < t < 8$. Therefore, $P(t)$ reaches its maximum at $t = 3$. The line is longest at $t = 3$; at that time there are
$700 + \int_0^3 r(t)\,dt - 3\cdot 800 = 700 + 3200 - 2400 = 1500$ people in the line.

(d) $700 + \int_0^t \left(r(u) - 800\right) du = 0$.

Question AB-4 [1]

(a) Area = $\int_0^9 6 - 2\sqrt{x}\,dx = \left(6x - \frac{4}{3}x^{3/2}\right)\Big|_0^9 = 54 - \frac{4}{3}\cdot 27 = 18$.

(b) Volume = $\pi\int_0^9 \left(\left(7 - 2\sqrt{x}\right)^2 - 1\right) dx$.

(c) Volume = $\int_0^6 \left(x \cdot 3x\right) dy = \int_0^6 \left(\frac{3y^4}{16}\right) dy$. [2]

Notes:

1. This is the second consecutive year that an area-volume problem has appeared in Part B, the closed calculator part, of the free response. Concern over calculator programs that can unfairly assist students in the solution of area-volume problems is one possible reason for this.
2. Not $\int_0^9 3\left(6 - 2\sqrt{x}\right)^2\,dx$. The sections are perpendicular to the <u>*y*-axis</u>, not the *x*-axis.

Question AB-5

(a) $g(x) = 5 + \int_0^x g'(t)\,dt$, so $g(3) = 5 + \int_0^3 g'(t)\,dt = 5 + \pi + \frac{3}{2}$ [1] and
$g(-2) = 5 + \int_0^{-2} g'(t)\,dt = 5 - \pi$. [2]

(b) $x = 0$ $x = 2$, and $x = 3$, because $g'(x)$ changes from increasing to decreasing or vice-versa at these points.

(c) At a critical point, $h'(x) = g'(x) - x = 0 \;\Rightarrow\; g'(x) = x$. The line $y = x$ intersects the semicircle where $x > 0$ and $x^2 + y^2 = 2x^2 = 4 \;\Rightarrow\; x = \sqrt{2}$. At that point, $h'(x)$ changes sign from positive to negative, so $h(x)$ has a relative maximum there.

The line $y = x$ also intersects the graph of $y = g'(x)$ at $x = 3$. $h'(x)$ does not change sign at $x = 3$, so there is neither a minimum nor a maximum there.

Notes:

1. Use geometry to calculate the areas, not the Fundamental Theorem!

2. The integral from 0 to -2 is negative.

Question AB-6

(a) At the point $(1, 2)$, $\frac{dy}{dx} = 8$. An equation of the tangent line is $y - 2 = 8(x - 1)$.

(b) The approximation is $y = 2 + 8 \cdot 0.1 = 2.8$. Since $\frac{d^2y}{dx^2} > 0$ on the open interval (1, 1.1), the graph of f is concave up there, and the tangent line lies below the graph of $y = f(x)$ for $1 \le x \le 1.1$. Thus the approximation is less than $f(1.1)$.

(c) $\int \frac{1}{y^3}\, dy = \int x\, dx \;\Rightarrow\; -\frac{1}{2y^2} = \frac{1}{2}x^2 + C$. Substituting $(1, 2)$ we get $-\frac{1}{8} = \frac{1}{2} + C \;\Rightarrow$ $C = -\frac{5}{8}$. Solving for y, $\frac{1}{y^2} = -x^2 + \frac{5}{4} \;\Rightarrow\; y = \left(\frac{5}{4} - x^2\right)^{-1/2}$ or $y = -\left(\frac{5}{4} - x^2\right)^{-1/2}$. Since $y(1) = 2 > 0$, the particular solution we are looking for is

$$y = \left(\frac{5}{4} - x^2\right)^{-1/2} = \frac{1}{\sqrt{\frac{5}{4} - x^2}}.$$

2010 BC
AP Calculus Free-Response Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

See AB Question 2.

Question BC-3

(a) $\dfrac{dx}{dt} = 2t - 4 \;\; \Rightarrow \left.\dfrac{dx}{dt}\right|_{t=3} = 2$. $\left.\dfrac{dy}{dt}\right|_{t=3} = \left.\left(te^{t-3} - 1\right)\right|_{t=3} = 2$. The speed at $t = 3$ is $\sqrt{2^2 + 2^2} = \sqrt{8}$.

(b) Total distance traveled = $\int_0^4 \sqrt{\left(2t-4\right)^2 + \left(te^{t-3} - 1\right)^2}\, dt \approx 11.588$.

(c) The tangent to the path is horizontal when $\dfrac{dy}{dt} = 0$. Solving gives $t \approx 2.208$. At that time, $\dfrac{dx}{dt} > 0$, so the particle is moving to the right.

(d)

(i) $x\left(t\right) = 5 \;\; \Rightarrow \;\; t^2 - 4t + 8 = 5 \;\; \Rightarrow \;\; t = 1$ or $t = 3$.

(ii) The slope is $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} =$. At $t = 1$, $\dfrac{dy}{dx} = \dfrac{1/e^2 - 1}{-2} = \dfrac{1}{2} - \dfrac{1}{2e^2}$.
At $t = 3$, $\dfrac{dy}{dx} = \dfrac{2}{2} = 1$.

(iii) $y\left(3\right) = y\left(2\right) + \int_2^3 \left(te^{t-3} - 1\right) dt = 3 + \dfrac{1}{e} + \int_2^3 \left(te^{t-3} - 1\right) dt \approx 4.000$. [1], [2]

Notes:

1. Alternatively, we could calculate $y(1)$.
2. The integral can be evaluated precisely using integration by parts:
$$\int_2^3 \left(te^{t-3} - 1\right) dt = \int_2^3 te^{t-3} dt - 1 = \left[te^{t-3} - \int_2^3 e^{t-3} dt\right]_2^3 - 1 = \left[(t-1)e^{t-3}\right]_2^3 - 1 = 1 - \frac{1}{e}.$$

Question BC-4

See AB Question 4.

Question BC-5

(a) The step in Euler's method is $-\frac{1}{2}$. At $(1, 0)$, the slope $\frac{dy}{dx} = 1 - y = 1$, so $y_{new} = 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$. At $\left(\frac{1}{2}, -\frac{1}{2}\right)$, the slope is $1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$, so $y_{new} = -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2} - \frac{3}{4} = -\frac{5}{4}$.

(b) Since *f* is continuous, $\lim_{x \to 1} f(x) = f(1) = 0$, so we can use l'Hôpital's Rule: $\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{1}{3}$.

(c) $\int \frac{1}{1-y} dy = \int dx \;\Rightarrow\; -\ln|1-y| = x + C$. Using the initial condition, we get $C = -1 \;\Rightarrow\; -\ln|1-y| = x - 1 \;\Rightarrow\; |1-y| = e^{1-x}$. We are told that $y = f(x) < 1$ for this solution, so $|1-y| = 1-y \;\Rightarrow\; 1 - y = e^{1-x} \;\Rightarrow\; y = 1 - e^{1-x}$.

Question BC-6

(a) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - ... + (-1)^n \dfrac{x^{2n}}{(2n)!} + ...$

$f(x) = \dfrac{\cos x - 1}{x^2} = -\dfrac{1}{2!} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} - ... + (-1)^n \dfrac{x^{2n-2}}{(2n)!} + ...$

(b) $f'(0) = 0$. $\dfrac{f''(0)}{2!} = \dfrac{1}{4!} \Rightarrow f''(0) > 0$. Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

(c) Integrating the first three terms of the series for f, we get

$P_5(x) = -\dfrac{1}{2!}x + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!} + C$. Since $g(0) = 1$, $C = 1 \Rightarrow$

$P_5(x) = 1 - \dfrac{1}{2!}x + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$.

(d) $g(1) = 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} - \dfrac{1}{5 \cdot 6!} + ...$ and $P_3(1) = 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!}$. The alternating series estimate error $|g(1) - P_3(1)|$ does not exceed the absolute value of the first omitted term in the series, which is $\dfrac{1}{5 \cdot 6!}$. Therefore, $|g(1) - P_3(1)| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

2010 AB (Form B)
AP Calculus Free-Response
Solutions and Notes

Question AB-1 (Form B)

(a) Area = $\int_0^2 \left(6-4\ln(3-x)\right)dx \approx 6.817$.

(b) Volume = $\pi\int_0^2 \left(\left(8-4\ln(3-x)\right)^2-2^2\right)dx \approx 168.180$.

(c) $\int_0^2 \left(6-4\ln(3-x)\right)^2 dx \approx 26.267$.

Question AB-2 (Form B)

(a) The graph of g has a horizontal tangent where $g'(x)=0$: $x = 0.163$ and $x = 0.359$.

(b) The graph of g is concave down where $g''(x)<0$. This happens on one subinterval, $0.129 < x < 0.223$. [1]

(c) $g(0.3)=g(1)+\int_1^{0.3} g'(x)\,dx \approx 1.546$. The slope at $x = 0.3$ is $g'(0.3) \approx -0.472$. An equation of the tangent line is $y-1.546=-0.472(x-0.3)$.

(d) Since $g''(x)>0$ for $0.25 < x < 1$, the graph of g is concave up there, so the tangent line at $x = 0.3$ lies under the graph of g.

Notes:

1. Since $g''(x)<0$ on (0.129, 0.223), we could say the graph of g is concave down on the <u>closed</u> interval [0.129, 0.223].

Question AB-3 (Form B)

(a) The midpoint Riemann sum is $4\cdot(46+57+62)=660$ cubic feet.

(b) The amount of water leaked = $\int_0^{12} R(t)\,dt \approx 225.594$ cubic feet.

(c) Volume = $1000+660-225.594=1434.406\approx 1434$ cubic feet.

(d) Let $V(t)$ be the volume of water in the pool at time *t*. Then

$V'(8)=P(8)-R(8)\approx 60-16.758=43.242$ cubic feet per hour. [1]

$V=\pi r^2 h=144\pi h$, so $V'(t)=144\pi\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt}=\dfrac{V'(t)}{144\pi}$. Therefore,

$\left.\dfrac{dh}{dt}\right|_{t=8}\approx\dfrac{43.242}{144\pi}\approx 0.096$ [1] feet per hour. [2]

Notes:

1. Or leave the answer as a fraction with π.

2. Don’t forget the units.

Question AB-4 (Form B)

(a) The squirrel changes direction at $t = 9$ and at $t = 15$, because its velocity changes sign at each of these points in time.

(b) The area of the trapezoid extending from $t = 0$ to $t = 9$ is $\frac{9+5}{2}\cdot 20 = 140$, so the squirrel moves 140 units towards *B* during that time. The area of the trapezoid from $t = 9$ to $t = 15$ is $\frac{6+4}{2}\cdot 10 = 50$, so the squirrel moves back 50 units towards *A* during that time. The area of the trapezoid from $t = 15$ to $t = 18$ is $\frac{3+2}{2}\cdot 10 = 25$, so the squirrel moves 25 units towards *B* during that time. Therefore, the distances from *A* are 140 at $t = 9$, 90 at $t = 15$, and 115 at $t = 18$. The maximum is 140 at $t = 9$.

(c) Based on the calculations in Part (b), the total distance traveled by the squirrel is $140 + 50 + 25 = 215$.

(d) $a(t) = \frac{-10-20}{10-7} = -10$; $v(t) = -10(t-9) = -10t + 90$; [1]
$x(t) = \int v(t)dt = -5t^2 + 90t + C$. From Part (b), $x(9) = 140 \;\Rightarrow$ [2]
$-5\cdot 9^2 + 90\cdot 9 + C = 140 \;\Rightarrow\; C = -265 \;\Rightarrow\; x(t) = -5t^2 + 90t - 265$.

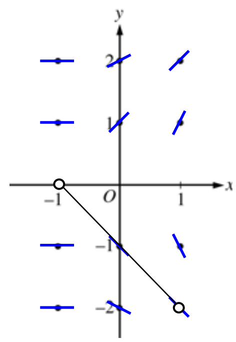
Notes:

1. The graph of velocity on the interval $7 < t < 10$ is a straight line. Its slope is equal to the acceleration. The equation of the line is written in the point-slope form for the point (9, 0). We could use the point (7, 20) instead: $v(t) = 20 - 10(t-7)$.

2. Be careful not to confuse $v(9) = 0$ on the graph, which applies to the equation for the <u>velocity</u>, with $x(9) = 140$, which we use to find the equation for the <u>distance</u>.

Question AB-5 (Form B)

(a)



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(b) $\frac{dy}{dx} = -1$ when $\frac{x+1}{y} = -1$, that is, $y = -x - 1$ and $y \neq 0$.

(c) $\int y\,dy = \int (x+1)\,dx$, so $\frac{y^2}{2} = \frac{x^2}{2} + x + C$. $y(0) = -2 \Rightarrow C = 2 \Rightarrow$
$y^2 = x^2 + 2x + 4 \Rightarrow |y| = \sqrt{x^2 + 2x + 4}$. Since the initial condition has $y < 0$, we choose $y = -\sqrt{x^2 + 2x + 4}$.

Notes:

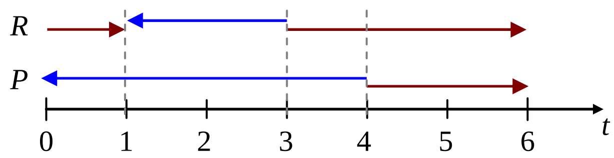
1. We are asked to sketch the solution on $-1 < x < 1$. Without this restriction, the domain of this solution would be $(-1, \infty)$. A particular solution to a differential equation is always defined on a connected interval. This differential equation is not defined when $y = 0$, so no solution can cross or touch the x-axis.

Question AB-6 (Form B)

(a) Particle R moves to the right when $r'(t) = 3t^2 - 12t + 9 > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow$ $0 \le t < 1$ and $3 < t \le 6$.

(b) Particle P moves to the right when $p'(t) = -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right) > 0 \Rightarrow \sin\left(\frac{\pi}{4}t\right) < 0 \Rightarrow$

$\pi < \frac{\pi}{4}t < 2\pi \Rightarrow 4 < t \le 6$.



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The particles travel in opposite directions for $0 < t < 1$ and $3 < t < 4$.

(c) The acceleration is $p''(3) = -\frac{\pi^2}{8}\cos\left(\frac{3\pi}{4}\right) = -\frac{\pi^2}{8}\cdot\left(-\frac{\sqrt{2}}{2}\right) = \frac{\pi^2\sqrt{2}}{16}$. The velocity at $t = 3$ is $p'(3) = -\frac{\pi}{2}\sin\left(\frac{3\pi}{4}\right) = -\frac{\pi\sqrt{2}}{4} < 0$. Since the velocity and acceleration have opposite signs, the particle is slowing down.

(d) The distance between the particles is $|p(t) - r(t)|$ and the average distance on the interval $1 \le t \le 3$ is $\frac{1}{3-1}\int_1^3 |p(t) - r(t)|\, dt$.

Notes:

1. A chart is not required but might be helpful.

2010 BC (Form B)
AP Calculus Free-Response
Solutions and Notes

Question BC-1 (Form B)

See AB Question 1.

Question BC-2 (Form B)

(a) The tangent line is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.[1] This occurs at $t \approx 1.145$ and $t \approx 1.253$.

(b) Let the position of the particle be $\left(x(t), y(t)\right)$. Then
$x(1) = x(0) + \int_0^1 14\cos\left(t^2\right)\sin\left(e^t\right)dt \approx -2 + 11.315 = 9.315$ and
$y(1) = y(0) + \int_0^1 1 + 2\sin\left(t^2\right)dt \approx 3 + 1.621 = 4.621$. At t = 1,
$\left.\frac{dy}{dx}\right|_{t=1} = \left.\left[\frac{dy}{dt}\Big/\frac{dx}{dt}\right]\right|_{t=1} \approx \frac{2.6829}{3.1072} \approx 0.863$. An equation for the tangent line is
$y - 4.621 = 0.863\left(x - 9.315\right)$.

(c) The speed is $\left.\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right|_{t=1} = \sqrt{3.1072^2 + 2.6829^2} \approx 4.105$.

(d) The acceleration vector is the derivative of the velocity vector. At t = 1, this is $\left(-28.425, 2.161\right)$.

Notes:

1. Enter the given functions $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into your calculator and save them for the rest of the solution.

Question BC-3 (Form B)

See AB Question 3.

Question BC-4 (Form B)

See AB Question 4.

Question BC-5 (Form B)

(a) $g'(x) = \dfrac{4\left(1+4x^2\right) - 4x \cdot 8x}{\left(1+4x^2\right)^2} = \dfrac{4-16x^2}{\left(1+4x^2\right)^2}$. For $x > 0$, $g'(x) = 0$ when $x = \dfrac{1}{2}$. For $0 < x < \dfrac{1}{2}$, $g'(x) > 0$ and g is increasing; for $x > \dfrac{1}{2}$, $g'(x) < 0$ and g is decreasing. The absolute maximum value of g therefore occurs at $x = \dfrac{1}{2}$. It is $g\left(\dfrac{1}{2}\right) = \dfrac{2}{1+4 \cdot \frac{1}{4}} = 1$. g has no minimum on $(0, \infty)$.

(b) The area is given by the improper integral

$$\int_1^\infty \left(\frac{1}{x} - \frac{4x}{1+4x^2} \right) dx =$$

$$\lim_{b\to\infty} \left(\left(\ln x - \frac{1}{2} \ln\left(1+4x^2\right) \right) \Bigg|_1^b \right) = \lim_{b\to\infty} \left(\ln b - \frac{1}{2}\ln\left(1+4b^2\right) + \frac{1}{2}\ln 5 \right) =$$

$$\lim_{b\to\infty} \ln \frac{b}{\sqrt{1+4b^2}} + \frac{1}{2}\ln 5 = \lim_{b\to\infty} \ln \frac{1}{\sqrt{\frac{1}{b}+4}} + \frac{1}{2}\ln 5 = \ln\frac{1}{2} + \frac{1}{2}\ln 5 .$$ [1]

Notes:

1. Leave it at this to save time and avoid mistakes.

Question BC-6 (Form B)

(a) By the ratio test, the series converges when $\lim_{n\to\infty}\left|\dfrac{\dfrac{(2x)^{n+1}}{n}}{\dfrac{(2x)^n}{n-1}}\right|<1$. Evaluating the limit gives $|2x|<1$ or $-\dfrac{1}{2}<x<\dfrac{1}{2}$. At $x=-\dfrac{1}{2}$, the series is $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n(-1)^n}{n-1}=\sum_{n=2}^{\infty}\frac{1}{n-1}$. This is the harmonic series, which diverges. At $x=\dfrac{1}{2}$, the series is $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}$. This is an alternating series with terms decreasing by absolute value and approaching zero. Therefore, it converges by the Alternating Series Test. The interval of convergence is $-\dfrac{1}{2}<x\le\dfrac{1}{2}$.

(b) $y'=f'(x)=\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n 2n(2x)^{n-1}}{n-1}$, so $xy'=\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n n\cdot(2x)^n}{n-1}$ and $xy'-y=\displaystyle\sum_{n=2}^{\infty}\left(\frac{(-1)^n n\cdot(2x)^n}{n-1}-\frac{(-1)^n(2x)^n}{n-1}\right)$. Factoring gives $xy'-y=\displaystyle\sum_{n=2}^{\infty}\left(\frac{(-1)^n(2x)^n(n-1)}{n-1}\right)=\sum_{n=2}^{\infty}(-2x)^n$. This is a geometric series with the first term $4x^2$ and the common ratio $-2x$, so it converges to $\dfrac{4x^2}{1+2x}$ when $|2x|<1$, that is, for $-\dfrac{1}{2}<x<\dfrac{1}{2}$.