Be Prepared

for the

AP

Calculus Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2005 AB

AP Calculus Free-Response
Solutions and Notes

Question 1

Solving $f(x) = g(x)$, the graphs intersect at $x = .17822$ and $x = 1$. Let $a = .17822$ and $b = 1$.[1]

(a) $A_R = \int_0^a g(x) - f(x)\,dx \approx 0.065$.[2,3]

(b) $A_S = \int_a^b f(x) - g(x)\,dx \approx 0.410$.

(c) $V_{y=-1} = \pi \int_a^b \left(1 + f(x)\right)^2 - \left(1 + g(x)\right)^2 dx \approx 4.559$.

Notes:

1. Store the intersection points in calculator variables and use those variables when calculating the integrals. See *Be Prepared*, page 256.

2. Refer to the functions as $f(x)$ and $g(x)$ in your answers to avoid transcription errors.

3. Use your calculator to evaluate the integrals; don't bother trying to antidifferentiate.

Question 2

(a) $\int_0^6 R(t)\,dt \approx 31.816$ cubic yards.[1]

(b) $Y(t) = 2500 + \int_0^t S(x) - R(x)\,dx$

(c) $Y'(t) = S(t) - R(t)$

$Y'(4) = S(4) - R(4) \approx -1.909$ cubic yards per hour.[1]

(d) Solving $Y'(t) = 0$, $t \approx 5.118$ hours. For $0 < t < 5.118$, $Y'(t) < 0$. For $5.118 < t < 6$, $Y'(t) > 0$. Thus, $Y(t)$ has a minimum at $t = 5.118$ of $Y(5.118) \approx 2492.369$ cubic yards. Since $Y(t)$ is continuous on [0, 6] and this is the only minimum, it is the absolute minimum.[2]

Notes:

1. Be attentive to the units.
2 Alternatively, you could also use the Candidate Test and evaluate $Y(t)$ at $t = 0$, $t = 5.118$, and $t = 6$. $Y(5.118)$ is the smallest of these.

Question 3

(a) $T'(7) \approx \dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}$ °C/cm.

(b) Average temperature = $\dfrac{1}{8}\int_0^8 T(x)\,dx$

$$\approx \frac{1}{8}\left(\frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2\right)$$ [1,2] $= 75.6875$ °C.

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = -45$ °C. This represents the change in temperature of the wire from the heated end to the other end.

(d) No. Since T is continuous and differentiable, by the MVT,[3] $T'(a) = \dfrac{93-70}{1-5} = -\dfrac{23}{4}$ for some a in (1, 5). Similarly, $T'(b) = \dfrac{70-62}{5-6} = -8$ for some b in (5, 6). Since T' is also continuous and differentiable, by the MVT for T', $T''(x) = \dfrac{-8-\left(-\frac{23}{4}\right)}{b-a} < 0$ for some x in (a, b).[4]

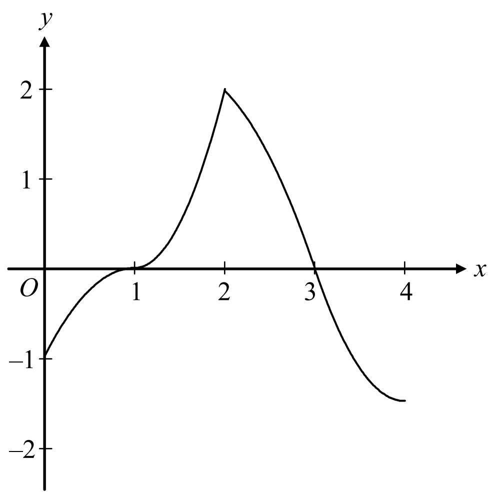
Notes:

1. Be careful with the unequal widths of the subintervals.
2. You can stop here — you are not required to simplify the answer.
3. Mean Value Theorem (it is OK to use this abbreviation).
4. Alternatively, we can say, Since $b > a$ and $T'(b) < T'(a)$, T' cannot be increasing everywhere on (a, b), so T'' cannot be positive everywhere on (a, b).

Question 4

(a) f has a local maximum at $x = 2$, since f' changes sign from positive to negative there.

(b)



[1]

(c) At $x = 1$, g has a relative minimum since $g'(x) = f(x)$ changes sign from negative to positive there .[2] At $x = 3$, g has a relative maximum since $g'(x) = f(x)$ changes sign from positive to negative there.

(d) The graph of g has a point of inflection at $x = 2$ since $g''(x) = f'(x)$ changes sign from positive to negative there.

Notes:

1. Negative, increasing, concave down for $0 < x < 1$; horizontal tangent at (1, 0); positive, increasing, concave up for $1 < x < 2$; corner at (2, 2); positive, decreasing, concave down for $2 < x < 3$; inflection point at (3, 0); negative, decreasing, concave up for $3 < x < 4$.

2. A sign chart is not enough for justification. You must explain in words the causal connection between the sign change of the derivative and the type of extremum.

Question 5

(a) $\int_0^{24} v(t)\,dt$ is equal to the area of the trapezoid with bases 12 and 24 and height 20.

Thus, $\int_0^{24} v(t)\,dt = \frac{12+24}{2}\cdot 20 = 360$ meters. This integral represents the total distance traveled by the car from $t = 0$ to $t = 24$ seconds (because the velocity was always non-negative over that time interval).

(b) $v'(20) = -\frac{20}{8}$ m/s^2. $v'(4)$ does not exist. $\lim\limits_{t\to 4^-} \frac{v(t)-v(4)}{t-4} = 5$ while

$\lim\limits_{t\to 4^+} \frac{v(t)-v(4)}{t-4} = 0$. Since the two one-sided limits are different,

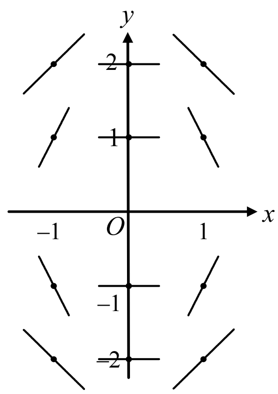
$f'(4) = \lim\limits_{t\to 4} \frac{v(t)-v(4)}{t-4}$ does not exist.

(c) $$a(t) = \begin{cases} 5, & \text{for } 0 < t < 4 \\ 0, & \text{for } 4 < t < 16 \\ -\frac{5}{2}, & \text{for } 16 < t < 24 \end{cases}$$

(d) Average rate of change of v on [8, 20] is $\frac{v(20)-v(8)}{20-8} = \frac{10-20}{12} = -\frac{5}{6}$. The MVT <u>does not</u> apply to v on [8, 20] because v is not differentiable at $t = 16$.

Question 6

(a)



(b) $y+1=2(x-1)$; $f(1.1)\approx 2(1.1-1)-1=-0.8$. [1]

(c) $\dfrac{dy}{dx}=-\dfrac{2x}{y} \Rightarrow y\,dy=-2x\,dx \Rightarrow \int y\,dy=\int -2x\,dx \Rightarrow \dfrac{y^2}{2}=-x^2+C$.

$\dfrac{1}{2}=-1+C \Rightarrow C=\dfrac{3}{2}$.

$y^2=-2x^2+3$; $y(1)=-1 \Rightarrow y=f(x)=-\sqrt{-2x^2+3}$. [2]

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Notes:

1. Make sure your answer is reasonably close to –1.
2. Choose the negative square root on the right side of the equation, since $y(1)=-1$.

2005 BC
AP Calculus Free-Response
Solutions and Notes

Question 1

See AB Question 1.

Question 2

(a) $Area = \dfrac{1}{2}\int_0^{\pi} \left(\theta + \sin\left(2\theta\right)\right)^2 \, d\theta \approx 4.382$.

(b) $x = r\cos\theta = -2 \;\Rightarrow\; \left(\theta + \sin\left(2\theta\right)\right)\cos\theta = -2$. Solving with the calculator gives $\theta \approx 2.786$.

(c) $\dfrac{dr}{d\theta} < 0$ for $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}$ means that r decreases as θ increases through values in this interval. Graphically this means that, since r is positive, the distance from the curve to the origin is getting smaller.[1]

(d) Since r is positive in the first quadrant, the distance from the origin $|r| = r$. Solving $\dfrac{dr}{d\theta} = 0$ gives $1 + 2\cos(2\theta) = 0 \;\Rightarrow\; \theta = \dfrac{\pi}{3}$. For $0 \le \theta < \dfrac{\pi}{3}$, $\dfrac{dr}{d\theta} > 0$ and for $\dfrac{\pi}{3} < \theta \le \dfrac{\pi}{2}$, $\dfrac{dr}{d\theta} < 0$. Since the derivative changes sign from positive to negative at $\theta = \dfrac{\pi}{3}$, $r(\theta)$ has a local maximum at $\theta = \dfrac{\pi}{3}$. r is a continuous function on the closed interval $\left[0, \dfrac{\pi}{2}\right]$ and this is the only maximum, so the absolute maximum is reached at $\theta = \dfrac{\pi}{3}$.[1]

Notes:

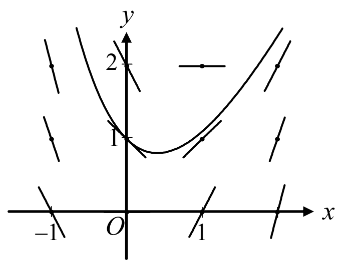
1. You can't readily see that because the scales on the *x*- and *y*-axes are different, and the graph is stretched vertically.
2. You could also use the Candidate Test, evaluating $r(\theta)$ at $\theta = 0$, $\theta = \dfrac{\pi}{3}$, and $\theta = \dfrac{\pi}{2}$ and selecting the largest of these, $r\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$.

Question 3

See AB Question 3.

Question 4

(a)



(b) Since there is a local minimum on the solution at $x = \ln\left(\frac{3}{2}\right)$ and $\frac{dy}{dx}$ exists there, $\frac{dy}{dx}$ must be zero at $x = \ln\left(\frac{3}{2}\right)$. Therefore,

$$2x - y = 2\ln\left(\frac{3}{2}\right) - y = 0 \Rightarrow y = 2\ln\left(\frac{3}{2}\right).$$

(c) $y_{new} = 1 + -1(-0.2) = 1.2$, so the second point is $(-0.2,\ 1.2)$. From here, $y_{new} = 1.2 + (-1.6)(-0.2) = 1.52 \Rightarrow f(-0.4) \approx 1.52$.[1]

(d) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$[2] $= 2 - (2x - y) = 2 - 2x + y$.

The Euler's Method approximation is less than $f(-0.4)$ because $\frac{d^2y}{dx^2} > 0$ over the interval $-0.4 \le x \le 0$. Indeed, $y'(0) = -1 < 0$ and y' is continuous and does not have any zeroes in the second quadrant, so y' remains negative there and $y''(x) = 2 - y' > 0$. Therefore the curve is concave up and the tangent lines will lie below the curve.

Notes:

1. Or draw and fill a table:

	Point 0	Point 1	Point 2
x	0	–0.2	–0.4
y	1	1 + (–1)(–0.2) = 1.2	1.2 + (–1.6)(–0.2) = 1.52
$m = 2x - y$	–1	2(–0.2) – 1.2 = –1.6	

2. The question says, Find in terms of x and y, so keep going: you must substitute for $\frac{dy}{dx}$.

Question 5

See AB Question 5.

Question 6

(a) $T_{6,x=2} = 7 + \frac{1}{3^2} \cdot \frac{(x-2)^2}{2!} + \frac{3!}{3^4} \cdot \frac{(x-2)^4}{4!} + \frac{5!}{3^6} \cdot \frac{(x-2)^6}{6!}$. [1]

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{2n \cdot 3^{2n}}$.

(c) Using the Ratio Test:[2] $\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{2n+2}}{(2n+2)\cdot 3^{2n+2}}}{\frac{(x-2)^{2n}}{2n \cdot 3^{2n}}} \right| = \lim_{n\to\infty} \left| \frac{2n(x-2)^2}{(2n+2)\cdot 3^2} \right| = \left| \left(\frac{x-2}{3} \right)^2 \right| < 1$.

Solving gives $-1 < x < 5$.

At $x = -1$[3] the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \sum_{n=1}^{\infty} \frac{1}{2n} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it contains the harmonic series. At $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}}$, which is the same as above and also diverges. Therefore, the interval of convergence is $-1 < x < 5$.

Notes:

1. It is easy to forget the factorial in the denominator, since the formula for the *n*th derivative already involves a factorial.
2. Don't forget the limit, and the absolute values.
3. The endpoints must be tested since the question asks about the interval of convergence, not just the radius.

2005 AB (Form B)
AP Calculus Free-Response
Solutions and Notes

Question 1

Solving $f(x) = g(x)$, the graphs intersect at $x = 0$ and $x = 1.1357$. Let $a = 1.1357$.[1]

(a) $A_R = \int_0^a f(x) - g(x)\,dx \approx 0.429$.[2,3]

(b) $V_{y=0} = \pi \int_0^a \left[f(x)\right]^2 - \left[g(x)\right]^2 dx \approx 4.267$.

(c) $V_{solid} = \dfrac{\pi}{2}\int_0^a r^2\,dx = \dfrac{\pi}{2}\int_0^a \left[\dfrac{f(x)-g(x)}{2}\right]^2 dx \approx 0.078$.

Notes:

1. Store the intersection points in calculator variables and use those variables when calculating the integrals. See *Be Prepared*, page 256.
2. Refer to the functions as $f(x)$ and $g(x)$ in your answers to avoid transcription errors. Also, store the functions as y_1 and y_2 and use these names when typing the integrals into your calculator.
3. Use your calculator to evaluate the integrals; don't bother trying to antidifferentiate.

Question 2

(a) No, because the rate of change in the amount of water at time $t = 15$ is $W(15) - R(15) = -121.09$ gal/hr.[1]

(b) The amount of water in the tank at time t is $A(t) = 1200 + \int_0^t W(t) - R(t)\,dt$. At $t = 18$, $A(18) = 1200 + \int_0^{18} W(t) - R(t)\,dt \approx 1309.788 \approx 1310$ gal.

(c) The rate of change in the amount of water is $A'(t) = W(t) - R(t) = 0$ when $t = 6.495$ hr. $A'(t)$ changes from negative to positive at $t = 6.495$, so there is a local minimum at $t = 6.495$.[2] Use the Candidate Test and evaluate $A(t)$, the total water in the tank, at $t = 0$, $t = 6.495$, and $t = 18$: $A(0) = 1200$, $A(6.495) = 525.242$, $A(18) = 1309.788$. $A(6.495)$ is the smallest of these, so the absolute minimum is at $t = 6.495$.

(d) $1309.788 - \int_{18}^{k} R(t)\,dt = 0$

Notes:

1. Be attentive to the units.
2. A sign chart for the derivative, by itself, is no longer sufficient for justification of an extremum. You need to state the causal connection between the derivative's sign change and the type of extremum.

Question 3

(a) $a(4) = v'(4) \approx 0.714$.

(b) The particle changes direction when $v(t)$ changes sign, at $t = 1$ and $t = 2$. The particle moves to the left for $1 < t < 2$ since $v(t) < 0$ over that time interval.

(c) $x(2) = 8 + \int_0^2 v(t)\,dt \approx 8.369$.

(d) Average speed $= \dfrac{1}{2-0}\int_0^2 \left|v(t)\right| dt \approx 0.371$.[1, 2]

Notes:

1. Speed equals the absolute value of the velocity of the particle.
2. You could also evaluate $\dfrac{1}{2-0}\left(\int_0^1 v(t)\,dt - \int_1^2 v(t)\,dt\right)$.

Question 4

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\dfrac{15}{2}$

$g'(-1) = f(-1) = -2$.

$g''(-1)$ would be $f'(-1)$, which does not exist, since the graph of *f* has a corner at $x = -1$.[1]

(b) There is a point of inflection at $x = 1$, since $g''(x) = f'(x)$ changes sign at $x = 1$ from positive to negative.

(c) $h(x) = 0$ for $x = -1$, $x = 1$, and $x = 3$.

(d) $h'(x) = -f(x) \Rightarrow h'(x) < 0$ where $f(x) > 0$, that is for $0 < x < 2$. Therefore, *h* is decreasing on the interval $0 \le x \le 2$.

Notes:

1. $\lim\limits_{x\to -1^-} \dfrac{f(x) - f(-1)}{x+1} = \dfrac{1}{3}$ while $\lim\limits_{x\to -1^+} \dfrac{f(x) - f(-1)}{x+1} = 2$

Question 5

(a) $y^2 = 2 + xy \Rightarrow 2y\dfrac{dy}{dx} = x\dfrac{dy}{dx} + y \Rightarrow (2y - x)\dfrac{dy}{dx} = y \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2} \Rightarrow 2y = 2y - x \Rightarrow x = 0$. When $x = 0,\ y^2 = 2 \Rightarrow y = \pm\sqrt{2}$. The points on the curve are $\left(0, \sqrt{2}\right)$ and $\left(0, -\sqrt{2}\right)$.

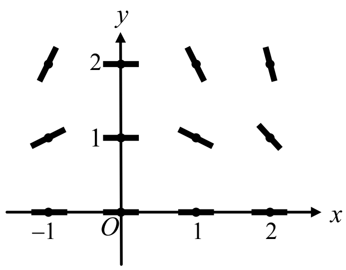
(c) For a tangent line to be horizontal, we must have $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{y}{2y - x} = 0 \Rightarrow y = 0$. But $y = 0$ and $y^2 = 2 + xy = 0$ gives $0 = 2$, which is not true. Thus, there are no horizontal tangent lines.

(d) If $y = 3$, then $x = \dfrac{7}{3}$ and $\dfrac{dy}{dx} = \dfrac{3}{2 \cdot 3 - \dfrac{7}{3}} = \dfrac{9}{11}$.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{\dfrac{dy}{dt}}{\dfrac{dy}{dx}} = \frac{6}{\dfrac{9}{11}} = \frac{22}{3}.$$

Question 6

(a)



(b) $f(-1)=2$ and $f'(x)\big|_{x=-1,y=2}=2$. Tangent line has equation $y-2=2(x+1)$.

(c) $\dfrac{dy}{dx}=\dfrac{-xy^2}{2} \Rightarrow \dfrac{dy}{y^2}=\dfrac{-x}{2}dx \Rightarrow \displaystyle\int\frac{dy}{y^2}=\int\frac{-x}{2}dx \Rightarrow -\frac{1}{y}=-\frac{x^2}{4}+C$.

$f(-1)=2 \Rightarrow -\dfrac{1}{2}=-\dfrac{1}{4}+C \Rightarrow C=-\dfrac{1}{4} \Rightarrow$

$-\dfrac{1}{y}=-\dfrac{x^2}{4}-\dfrac{1}{4} \Rightarrow \dfrac{1}{y}=\dfrac{x^2+1}{4} \Rightarrow y=\dfrac{4}{x^2+1}$.

2005 BC (Form B)
AP Calculus Free-Response Solutions and Notes

Question 1

(a) $\vec{a}(t)=\left(12-6t,\ \dfrac{4(t-4)^3}{1+(t-4)^4}\right)$; $\vec{a}(2)=\left(0,-\dfrac{32}{17}\right)\approx(0,\ -1.882)$.

Speed = $\left|\vec{v}(t)\right|=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$.

$\left|\vec{v}(2)\right|=\sqrt{12^2+(\ln 17)^2}\approx 12.330$.

(b) $y(2)=5+\int_0^2 \ln\left(1+(t-4)^4\right)dt \approx 13.671$.

(c) Slope = $\left.\dfrac{dy}{dx}\right|_{t=2}=\dfrac{\left.\dfrac{dy}{dt}\right|_{t=2}}{\left.\dfrac{dx}{dt}\right|_{t=2}}=\dfrac{\ln 17}{12}\approx 0.236$.

An equation of the tangent line is $y-13.671=0.236(x-3)$.[1]

(d) $\dfrac{dx}{dt}=3t(t-4)=0 \Rightarrow t=0$ or $t=4$. $\dfrac{dy}{dt}=\ln\left(1+(t-4)^4\right)=0 \Rightarrow t=4$. The object is at rest when $t=4$.

Notes:

1. Or $y-y(2)=\dfrac{\ln 17}{12}(x-3)$.

Question 2

See AB Question 2.

Question 3

(a) Since the graph of *f* has a horizontal tangent at $x = 0$, $f'(0) = 0$. From the given formula, $f''(0) = \dfrac{-3!}{5^2 \cdot 1^2} = -\dfrac{6}{25} < 0$. By the Second Derivative Test, *f* has a local maximum at $x = 0$.

(b) $T_{3,f} = 6 - \dfrac{3!}{5^2} \cdot \dfrac{x^2}{2!} + \dfrac{4!}{5^3 \cdot 2^2} \cdot \dfrac{x^3}{3!}$ [1], [2] $= 6 - \dfrac{3}{25}x^2 + \dfrac{1}{125}x^3$.

(c) The *n*th term of the Taylor series for *f* about $x = 0$ is given by $a_n = \dfrac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2 \cdot n!} \cdot x^n = \dfrac{(-1)^{n+1}(n+1)x^n}{5^n(n-1)^2}$. Using the Ratio Test to find the radius of convergence, we get: $\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{(n+2)x^{n+1}}{5^{n+1}n^2}}{\dfrac{(n+1)x^n}{5^n(n-1)^2}}\right| =$ [3] $\lim\limits_{n\to\infty}\left|\dfrac{(n+2)(n-1)^2}{(n+1)n^2} \cdot \dfrac{x}{5}\right| = \left|\dfrac{x}{5}\right| < 1 \Rightarrow$ $-5 < x < 5$, so the radius of convergence is 5.[4]

Notes:

1. Be careful: it is easy to forget the factorial in the denominator, since the formula for the *n*th derivative already involves a factorial.
2. You don't have to simplify.
3. Don't forget the limit, and the absolute values.
4. Don't test the endpoints since you are asked for the radius of convergence, not the interval.

Question 4

See AB Question 4.

Question 5

See AB Question 5.

Question 6

(a) $A_R = \int_0^k f(x)\,dx = \int_0^k \frac{1}{x+2}\,dx = \ln(x+2)\Big|_0^k = \ln(k+2) - \ln(2)$.

(b) $V_R = \pi\int_0^k \left(\frac{1}{x+2}\right)^2 dx = \pi\cdot\left(-(x+2)^{-1}\right)\Big|_0^k = \pi\cdot\left(\frac{1}{2} - \frac{1}{k+2}\right)$.

(c) $V_S = \pi\int_k^\infty \left(\frac{1}{x+2}\right)^2 dx = \pi\cdot\lim\limits_{a\to\infty}\left(-(x+2)^{-1}\right)\Big|_k^a = \pi\cdot\lim\limits_{a\to\infty}\left(\frac{1}{k+2} - \frac{1}{a+2}\right) = \pi\cdot\frac{1}{k+2}$.

$V_S = V_R \Rightarrow \pi\frac{1}{k+2} = \pi\left(\frac{1}{2} - \frac{1}{k+2}\right) \Rightarrow \frac{2}{k+2} = \frac{1}{2} \Rightarrow k = 2$.