Be Prepared

for the

AP

Calculus Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2004 AB
AP Calculus Free-Response
Solutions and Notes

Question 1

(a) $\int_0^{30}\left[82+4\sin\left(\frac{t}{2}\right)\right]dt \approx 2474$.

(b) $\left.\frac{dF}{dt}\right|_{t=7} \approx -1.873 < 0 \Rightarrow F(t)$ is decreasing.

(c) $\frac{1}{5}\int_{10}^{15}\left[82+4\sin\left(\frac{t}{2}\right)\right]dt \approx 81.899$ cars/min.

(d) $\frac{1}{5}\left[F(15)-F(10)\right]=\frac{1}{5}\left[4\sin\left(\frac{15}{2}\right)-4\sin\left(\frac{10}{2}\right)\right] \approx 1.518$ cars/min^2.

Question 2

(a) Area = $\int_0^1\left[f(x)-g(x)\right]dx=\int_0^1\left[2x(1-x)-3(x-1)\sqrt{x}\right]dx \approx 1.133$.

(b) Volume= $\pi\int_0^1\left[\left(2-g(x)\right)^2-\left(2-f(x)\right)^2\right]dx =$

$\pi\int_0^1\left[\left(2-3(x-1)\sqrt{x}\right)^2-\left(2-2x(1-x)\right)^2\right]dx \approx 16.179$.

(c) $\int_0^1\left(h(x)-g(x)\right)^2 dx=\int_0^1\left(kx(1-x)-3(x-1)\sqrt{x}\right)^2 dx = 15$.[1]

Notes:

1. Do not solve.

Question 3

(a) $a(t) = \dfrac{dv}{dt} = -\dfrac{e^t}{1+\left(e^t\right)^2} = -\dfrac{e^t}{1+e^{2t}} \Rightarrow a(2) = -\dfrac{e^2}{1+e^4}$.[1]

(b) $v(2) = 1 - \tan^{-1}\left(e^2\right) < 0$ and, from Part (a), $a(2) < 0$. Since $v(2)$ and $a(2)$ have the same sign, the speed is increasing.[2]

(c) $v(t) = 0 \Rightarrow \tan^{-1}\left(e^t\right) = 1 \Rightarrow e^t = \tan(1) \Rightarrow t = \ln\left(\tan(1)\right) \approx 0.443$. There is a local max at $t = .443$ since v is positive to the left and negative to the right. $t = .443$ is the only critical number in the domain, therefore y reaches an absolute maximum at this time.[3]

(d) $y(2) = -1 + \int_0^2 \left(1 - \tan^{-1}\left(e^t\right)\right) dt \approx -1.360$. The particle is moving away from the origin because $y(2) < y(0)$ and, $v(2) < 0$ (from Part (b)).

Notes:

1. Leave it at that to save time and avoid mistakes. Or just use your calculator to evaluate the derivative: $a(2) = v'(2) \approx -0.133$.
2. Give a reason for full credit.
3. Justification is required for full credit.

Question 4

(a) Using implicit differentiation,
$2x + 8yy' = 3y + 3xy' \;\Rightarrow\; y'(8y - 3x) = 3y - 2x \;\Rightarrow\; y' = \dfrac{3y - 2x}{8y - 3x}$.

(b) $\dfrac{dy}{dx} = 0 \;\Rightarrow\; 3y - 2x = 0$. $x = 3$ and $y = 2$ satisfy this equation, $8 \cdot 2 - 3 \cdot 3 \neq 0$, and[1] $3^2 + 4 \cdot 2^2 = 7 + 3 \cdot 3 \cdot 2$, so the point $P = (3, 2)$ is indeed on the curve.

(c) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{3y - 2x}{8y - 3x}\right] = \dfrac{(8y - 3x)\left(3\dfrac{dy}{dx} - 2\right) - (3y - 2x)\left(8\dfrac{dy}{dx} - 3\right)}{(8y - 3x)^2}$. At $P = (3, 2)$,
$3y - 2x = 0,\ 8y - 3x = 7$, and $\dfrac{dy}{dx} = 0 \;\Rightarrow\; \left.\dfrac{d^2y}{dx^2}\right|_{x=3,\,y=2} = -\dfrac{2}{7}$. The first derivative is 0 and the second derivative is negative, so the curve has a local maximum at P.

Notes:

1. You must verify that the point is on the curve.

Question 5

(a) $g(0) = \int_{-3}^{0} f(x)\,dx = 3 \cdot \dfrac{2+1}{2}$. $g'(0) = f(0) = 1$.

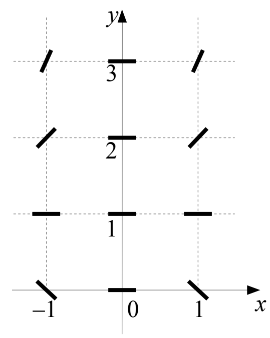
(b) At a relative maximum, $g'(x) = f(x)$ must change from positive to negative. There is only one such point, at $x = 3$.

(c) The absolute minimum may occur at the end points $x = -5$ or $x = 4$, or at $x = -4$ where $g'(x) = f(x)$ changes from negative to positive.
$g(-5) = 0,\ g(4) > 0$, and $g(-4) = -1$, so the answer is $g(-4) = -1$.

(d) At a point of inflection, the derivative changes from increasing to decreasing or vice-versa. There are three such points: at $x = -3,\ x = 1$, and $x = 2$.

Question 6

(a)



(b) $y > 1,\ x \neq 0$.

(c) $\dfrac{dy}{dx} = x^2(y-1) \Rightarrow \displaystyle\int \frac{dy}{y-1} = \int x^2 dx \Rightarrow \ln|y-1| = \frac{1}{3}x^3 + C$. $x = 0, y = 3 \Rightarrow C = \ln 2$.

$|y-1| = 2e^{\frac{x^3}{3}} \Rightarrow y = 1 + 2e^{\frac{x^3}{3}}$.

2004 BC
AP Calculus Free-Response
Solutions and Notes

Question 1

See AB Question 1.

Question 2

See AB Question 2.

Question 3

(a) $x(4) = x(2) + \int_2^4 x'(t)\,dt = 1 + \int_2^4 \left[3 + \cos\left(t^2\right)\right] dt \approx 7.133$.

(b) $y - y(2) = \dfrac{y'(2)}{x'(2)}\left(x - x(2)\right) \Rightarrow y - 8 = \dfrac{-7}{3 + \cos(4)}(x - 1)$. [1]

(c) Speed = $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = \sqrt{\left(3 + \cos(4)\right)^2 + (-7)^2}$. [1]

(d) $x'(t) = 3 + \cos\left(t^2\right) \Rightarrow x''(4) = \dfrac{d}{dt}\left[x'(t)\right]\Big|_{t=4} \approx 2.303$.

$\dfrac{y'(t)}{x'(t)} = 2t + 1 \Rightarrow y'(t) = (2t+1)\left[3 + \cos\left(t^2\right)\right] \Rightarrow y''(4) = \dfrac{d}{dt}\left[y'(t)\right]\Big|_{t=4} \approx 24.814$.

$\vec{a}(4) = \left(x''(4),\ y''(4)\right) = (2.303,\ 24.814)$.

Notes:

1. To avoid mistakes and save time, do not simplify.

Question 4

See AB Question 4.

Question 5

(a) This is a logistic model with carrying capacity 12, so for both initial conditions, $P(0)=3$ and $P(0)=20$, $\lim_{t\to\infty} P(t)=12$.

(b) P grows from 3 approaching 12. P is growing the fastest when P is half the carrying capacity, that is, $P = 6$.

(c) $\frac{dY}{dt}=\frac{Y}{5}\left(1-\frac{t}{12}\right) \Rightarrow \int\frac{dY}{Y}=\frac{1}{5}\int\left(1-\frac{t}{12}\right)dt \Rightarrow \ln|Y|=\frac{1}{5}\left(t-\frac{t^2}{24}\right)+C$.

$Y(0)=3 \Rightarrow C=\ln 3$. $\ln Y=\frac{1}{5}\left(t-\frac{t^2}{24}\right)+\ln 3 \Rightarrow Y(t)=3e^{\frac{1}{5}\left(t-\frac{t^2}{24}\right)}$.

(d) $\left(t-\frac{t^2}{24}\right)\to -\infty \Rightarrow \lim_{t\to\infty} Y(t)=\lim_{t\to\infty} 3e^{\frac{1}{5}\left(t-\frac{t^2}{24}\right)}=0$.

Question 6

(a) $f(0)=\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$; $f'(0)=5\cos\left(\frac{\pi}{4}\right)=\frac{5\sqrt{2}}{2}$;

$f''(0)=-25\sin\left(\frac{\pi}{4}\right)=-\frac{25\sqrt{2}}{2}$; $f'''(0)=-125\sin\left(\frac{\pi}{4}\right)=-\frac{125\sqrt{2}}{2}$.

$$P(x)=\frac{\sqrt{2}}{2}+\frac{5\sqrt{2}}{2}x-\frac{25\sqrt{2}}{2\cdot 2!}x^2-\frac{125\sqrt{2}}{2\cdot 3!}x^3.$$

(b) $\frac{f^{(22)}(x)}{22!}=-\frac{5^{22}\sqrt{2}}{2\cdot 22!}$. [1]

(c) $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|=\left|\frac{5^4\sin\left(5c+\frac{\pi}{4}\right)}{4!}\left(\frac{1}{10}\right)^4\right|<\frac{5^4}{4!\cdot 10^4}=\frac{1}{24\cdot 16}<\frac{1}{100}$.

(d) This polynomial can be obtained by integrating the first three terms in $P(x)$, and taking into account that $G(0)=0$. We get $\frac{\sqrt{2}}{2}x+\frac{5\sqrt{2}}{2\cdot 2}x^2-\frac{25\sqrt{2}}{2\cdot 6}x^3$.

Notes:

1. The sign for the k-th term goes +, +, –, –, etc., that is

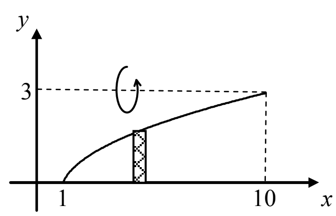
k	0	1	2	3	...	20	21	22
sign	+	+	–	–	...	+	+	–

2004 AB (Form B)
AP Calculus Free-Response
Solutions and Notes

Question 1

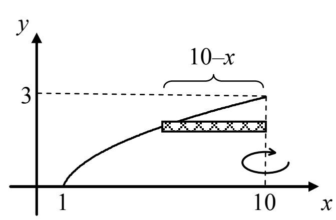
(a) Area = $\int_1^{10} \sqrt{x-1}\,dx = 18$.

(b)



Volume = $\pi\int_1^{10}\left[3^2-\left(3-\sqrt{x-1}\right)^2\right]dx \approx 212.058$.

(c)

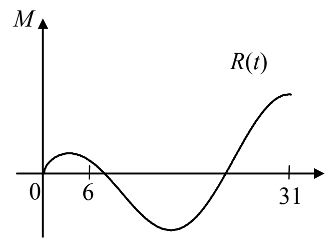


Volume = $\pi\int_0^{3}\left(9-y^2\right)^2\,dy$ [1] ≈ 407.150.

Notes:

1. $y=\sqrt{x-1} \Rightarrow x=y^2+1 \Rightarrow 10-x=9-y^2$

Question 2



(a) Increasing, because $R(6) = 5\sqrt{6}\cos\left(\frac{6}{5}\right) \approx 4.438 > 0$.

(b) Increasing at a decreasing rate, because $R'(6) \approx -1.913 < 0$.

(c) Number of mosquitoes = $1000 + \int_0^{31} 5\sqrt{t}\cos\left(\frac{t}{5}\right)dt \approx 964$.[1]

(d) $R(t) > 0$ for $0 < t < \frac{5\pi}{2}$, $R(t) < 0$ for $\frac{5\pi}{2} < t < \frac{15\pi}{2}$, and $R(t) > 0$ for $\frac{15\pi}{2} < t \le 31$.

Therefore the maximum could occur either at $t = \frac{5\pi}{2}$ or at $t = 31$. The number of mosquitoes at $t = \frac{5\pi}{2}$ is $1000 + \int_0^{\frac{5\pi}{2}} 5\sqrt{t}\cos\left(\frac{t}{5}\right)dt \approx 1039$. This is greater than 964, the number of mosquitoes at $t = 31$ (from Part (c)), so 1039 is the maximum.[2]

Notes:

1. Don't forget to round.
2. To keep things simple, you might as well take all the points where $R(t) = 0$ on $[0, 31]$, namely at $t = \frac{5\pi}{2}$ and $t = \frac{15\pi}{2}$, plus both endpoints, and compare the number of mosquitoes at these four times.

Question 3

(a) $\int_0^{40} v(t)\,dt \approx \frac{40}{4}\big(v(5)+v(15)+v(25)+v(35)\big) = 10\big(9.2+7.0+2.4+4.3\big) = 229$ miles.

$\int_0^{40} v(t)\,dt$ is the distance in miles traveled by the plane from 0 to 40 minutes.

(b) $v(0) = v(15)$ and $v(25) = v(30)$. By Rolle's theorem (or the Mean Value Theorem), acceleration, which is $v'(t)$, must equal 0 at least once on each of the intervals [0, 15] and [25, 30]. The answer is 2.

(c) The acceleration at $t = 23$ is $f'(23) \approx -0.408$ miles/min^2.

(d) Average velocity =

$$\frac{1}{40}\int_0^{40} f(t)\,dt = \frac{1}{40}\int_0^{40}\left[6+\cos\left(\frac{t}{10}\right)+3\sin\left(\frac{7t}{40}\right)\right]dt \approx 5.916 \text{ miles/min.}$$

Question 4

(a) f' changes from increasing to decreasing at $x = 1$ and from decreasing to increasing $x = 3$. Therefore, there are two points of inflection, at $x = 1$ and $x = 3$.

(b) $f(x) = \int_{-1}^{x} f'(t)dt + C$. Since $f'(t) \le 0$ for $-1 \le t \le 4$ and $f'(t) \ge 0$ for $4 \le t \le 5$, f reaches the absolute minimum at $x = 4$. f could reach the absolute maximum at $x = -1$ or $x = 5$. But $f(5) - f(-1) = \int_{-1}^{5} f'(t)dt < 0$ (because the area of the region below the x-axis for $-1 \le t \le 4$ appears much bigger than the area of the region above the x-axis for $4 \le t \le 5$). Therefore, the absolute maximum is at $x = -1$.

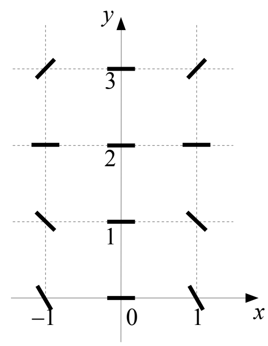
(c) $g(2) = 2f(2) = 12$. $g'(2) = \left(f(x) + xf'(x)\right)\Big|_{x=2}$ [1] $= 6 + 2(-1) = 4$. An equation for the tangent line is $y - g(2) = g'(2)(x-2) \Rightarrow y - 12 = 4(x-2)$.

Notes:

1. Using the Product Rule.

Question 5

(a)



(b) $x \neq 0, y < 2$.

(c) $\displaystyle\int \frac{dy}{y-2} = \int x^4 dx \Rightarrow \ln|y-2| = \frac{x^5}{5} + C$. $y(0) = 0 \Rightarrow C = \ln 2$.

$|y-2| = 2e^{\frac{x^5}{5}} \Rightarrow y = 2 - 2e^{\frac{x^5}{5}}$. [1]

Notes:

1. We reject the solution $y = 2 + 2e^{\frac{x^5}{5}}$ because for this solution $y(0) \neq 0$.

Question 6

(a) $\int_0^1 x^n\,dx = \left.\dfrac{x^{n+1}}{n+1}\right|_0^1 = \dfrac{1}{n+1}$.

(b) The equation for the tangent line at (1, 1) is
$y-1 = n\left(1^{n-1}\right)(x-1) \Rightarrow y = 1+n(x-1)$. Its x-intercept is $x = 1-\dfrac{1}{n}$. The area of the triangle is $\dfrac{1}{2}(1)\left(1-\left(1-\dfrac{1}{n}\right)\right) = \dfrac{1}{2n}$.

(c) Area of $S = \int_0^1 x^n\,dx -$ area of $T = g(n) = \dfrac{1}{n+1}-\dfrac{1}{2n}$. $g'(n) = -\dfrac{1}{(n+1)^2}+\dfrac{1}{2n^2}$.
$g'(n)=0$ and changes sign from positive to negative when
$(n+1)^2 = 2n^2 \Rightarrow n+1 = \sqrt{2}n \Rightarrow n = \dfrac{1}{\sqrt{2}-1}$. Therefore, the area of S reaches maximum at $n = \dfrac{1}{\sqrt{2}-1}$.[1]

Notes:

1. Do not despair if your answer is $\sqrt{2}+1$: $\dfrac{1}{\sqrt{2}-1} = \sqrt{2}+1$.

2004 BC (Form B)
AP Calculus Free-Response
Solutions and Notes

Question 1

(a) Speed $s(t)=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$. $s(0)=\sqrt{\left(t^4+9\right)+\left(2e^t+5e^{-t}\right)^2}\Big|_{t=0}=\sqrt{9+49}$.

$a(t)=\left\langle \frac{4t^3}{2\sqrt{t^4+9}}, 2e^t-5e^{-t}\right\rangle\Big|_{t=0}=\langle 0,-3\rangle$.

(b) $\left.\frac{dy}{dx}\right|_{t=0}=\left.\frac{2e^t+5e^{-t}}{\sqrt{t^4+9}}\right|_{t=0}=\frac{7}{3}$. An equation for the tangent line is $y-1=\frac{7}{3}(x-4)$.

(c) Distance = $\int_0^3 s(t)\,dt=\int_0^3\sqrt{\left(t^4+9\right)+\left(2e^t+5e^{-t}\right)^2}\,dt \approx 45.227$.

(d) $x(3)=x(0)+\int_0^3\frac{dx}{dt}dt=4+\int_0^3\sqrt{t^4+9}\,dt \approx 4+13.931$.

Question 2

(a) $f(2) = 7$. $\dfrac{f''(2)}{2!} = -9 \Rightarrow f''(2) = -18$.

(b) Yes. $f'(2) = 0$ and $f''(2) < 0$. Therefore $f(x)$ has a relative maximum at $x = 2$.

(c) $f(0) \approx T(0) = 7 - 9(-2)^2 - 3(-2)^3 = -5$. We do not have enough information to determine whether $x = 0$ is a critical number for $f(x)$. For example, the whole family of functions $g(x) = T(x) + C(x-2)^4$, where C is any constant, has the same third-degree Taylor polynomial $T(x)$. $f(x)$ could be any one of these functions. One of these functions has a critical number at $x = 0$ — when $g'(0) = -18 \cdot (-2) - 3 \cdot 3(-2)^2 + 4C(-2)^3 = 0$ — others don't.

(d) $f(0) - T(0) = \dfrac{f^{(4)}(c)}{4!}(-2)^4$ for some $0 \le c \le 2$. Therefore,

$$\left|f(0) - (-5)\right| \le \frac{6}{4!}2^4 = 4 \;\Rightarrow\; f(0) + 5 \le 4 \;\Rightarrow\; f(0) \le -1 < 0.$$

Question 3

See AB Question 3.

Question 4

See AB Question 4.

Question 5

(a) $\frac{1}{4-1}\int_1^4 \frac{dx}{\sqrt{x}} = \frac{1}{3}2\sqrt{x}\Big|_1^4 = \frac{2}{3}$.

(b) Volume = $\pi\int_1^4 \left(g(x)\right)^2 dx = \pi\int_1^4 \frac{1}{x}dx = \pi \ln 4$.

(c) Average area = $\frac{\text{Volume}}{4-1} = \pi\frac{\ln 4}{3}$.

(d) $\int_4^b g(x)\,dx = 2\sqrt{b} - 2\sqrt{4} \to \infty$, when $b \to \infty$.

$$\lim_{b\to\infty}\frac{\int_4^b g(x)\,dx}{b-4} = \lim_{b\to\infty}\frac{2\sqrt{b}-2\sqrt{4}}{b-4} = \lim_{b\to\infty}\frac{2}{\sqrt{b}+\sqrt{4}} = 0.$$ [1]

Notes:

1. Or use l'Hôpital's Rule: $\lim_{b\to\infty}\frac{2\sqrt{b}-2\sqrt{4}}{b-4} = \lim_{b\to\infty}\frac{\frac{1}{\sqrt{b}}}{1} = 0$

Question 6

See AB Question 6.