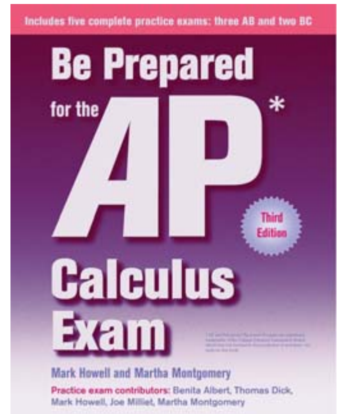

Be Prepared
for the
AP
Calculus Exam

Third Edition

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2018 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) $\int_0^{300} r(t)\,dt = 270$ people. [1]

(b) $20 + 270 - 0.7 \cdot 300 = 80$ people.

(c) Solve $290 - 0.7t = 0 \Rightarrow t \approx 414.286$ seconds.

(d) The number of people waiting in line at the escalator is given by $N(t) = 20 + \int_0^t r(u) - 0.7du$, for $0 \le t \le 300$. Since $N(t)$ is continuous, it must have an absolute minimum on $0 \le t \le 300$, and this minimum can occur only at an endpoint or critical point. $N'(t) = r(t) - 0.7 = 0 \Rightarrow t \approx 33.013$ and $t \approx 166.575$. $N(0) = 20$, $N(33.013) \approx 3.803$, $N(166.575) \approx 158.070$, and $N(300) = 80$. [2] The minimum number of people waiting at the escalator occurs at $t = 33.013$ seconds, and is approximately 4.

Notes:

1. Use the given name for the rate function in your setup, rather than copying its definition. This is faster and avoids transcription errors.

2. The simplest way to justify an extreme value of a continuous function on a closed interval is by the candidate test, illustrated here. Note that this test is a consequence of the Extreme Value Theorem.

Question AB-2

(a) The acceleration is $a(3) = v'(3) \approx -2.118$. [1]

(b) $x(3) = x(0) + \int_0^3 v(t)\,dt \approx -1.760$.

(c) $\int_0^{3.5} v(t)\,dt \approx 2.844$. This is the net change in the position, or displacement, of the particle from $t = 0$ to $t = 3.5$.
$\int_0^{3.5} \left|v(t)\right| dt \approx 3.737$. This is the total distance the particle travels from $t = 0$ to $t = 3.5$.

(d) $x_2'(t) = 2t - 1 = v(t) \;\Rightarrow\; t \approx 1.571$.

Notes:

1. When no units are given in the problem, don't waste time making up your own.

Question AB-3

(a) $g(x) = f'(x) \;\Rightarrow\; f(-5) = f(1) + \int_1^{-5} g(t)\,dt = 3 + \left(-1 + \frac{3}{2} + 9\right)$ [1] $= \frac{25}{2}$.

(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 2(x-4)^2\,dx = 4 + \frac{2}{3}\cdot(x-4)^3\Big|_3^6 = 4 + \frac{2}{3}(8+1)$ [1] $= 10$.

(c) f is increasing where $g(x) = f'(x)$ is non-negative and f is concave up where $g'(x) = f''(x)$ is positive. f is increasing and concave up on the intervals $0 < x < 1$ and $4 < x < 6$.

(d) A point of inflection on the graph of f occurs at $x = 4$, because the graph of $g(x) = f'(x)$ changes from decreasing to increasing there.

Notes:

1. You can leave it at this, no need to do the arithmetic.

Question AB-4

(a) $H'(6) \approx \dfrac{H(7)-H(5)}{7-5} = \dfrac{11-6}{2} = \dfrac{5}{2}$ meters per year. The height of the tree is increasing at the rate of approximately $\dfrac{5}{2}$ meters per year at time $t = 6$ years.

(b) On the interval $3 \le t \le 5$, the average rate of change of H is $\dfrac{6-2}{5-3} = 2$. Since H is differentiable on $3 < t < 5$ and continuous on $3 \le t \le 5$, the Mean Value Theorem guarantees there is at least one time t, $3 < t < 5$, such that $H'(t) = 2$. This time t is also in the interval $2 < t < 10$.

(c) $\dfrac{1}{8} \cdot \left(\dfrac{3.5}{2} \cdot 1 + \dfrac{8}{2} \cdot 2 + \dfrac{17}{2} \cdot 2 + \dfrac{26}{2} \cdot 3 \right)$ meters. [1]

(d) $G(x) = 50 \;\Rightarrow\; 50 = \dfrac{100x}{1+x} \Rightarrow 50 + 50x = 100x \;\Rightarrow\; x = 1$ meters.
$\dfrac{dG}{dt} = G'(x) \cdot \dfrac{dx}{dt} = \dfrac{(1+x) \cdot 100 - 100x}{(1+x)^2} \cdot \dfrac{dx}{dt} = \dfrac{100}{(1+x)^2} \cdot \dfrac{dx}{dt}$. When the diameter of base of the tree x is 1 meter, $\dfrac{dG}{dt} = \dfrac{100}{4} \cdot 0.03 = 0.75$ meters per year.

Notes:

1. For the record, this simplifies to $\dfrac{263}{32}$ meters, but the simplification is not needed to earn full credit.

Question AB-5

(a) $\dfrac{f(\pi)-f(0)}{\pi-0}=\dfrac{-e^{\pi}-1}{\pi}$.

(b) $f'(x)=-e^x\sin x+e^x\cos x=e^x(\cos x-\sin x)$. The slope of the tangent line at $x=\dfrac{3\pi}{2}$ is $f'\left(\dfrac{3\pi}{2}\right)=e^{\frac{3\pi}{2}}$.

(c) Since *f* is a continuous function, it must have an absolute minimum on the closed interval, $0\le x\le 2\pi$. The absolute minimum must occur at an endpoint or at a critical point. $f'(x)=0 \Rightarrow x=\dfrac{\pi}{4}$ or $x=\dfrac{5\pi}{4}$. Using the candidate test [1], we have

x	0	$\dfrac{\pi}{4}$	$\dfrac{5\pi}{4}$	2π
$f(x)$	1	$e^{\pi/4}\cdot\dfrac{\sqrt{2}}{2}$	$e^{5\pi/4}\cdot\left(-\dfrac{\sqrt{2}}{2}\right)$	$e^{2\pi}$

The absolute minimum is $e^{5\pi/4}\cdot\left(-\dfrac{\sqrt{2}}{2}\right)$ (the only negative number among the candidates).

(d) Since *f* and *g* are both continuous, $\lim\limits_{x\to\pi/2}f(x)=f\left(\dfrac{\pi}{2}\right)=0$ and

$\lim\limits_{x\to\pi/2}g(x)=g\left(\dfrac{\pi}{2}\right)=0$, [2] by L'Hospital's Rule

$$\lim_{x\to\pi/2}\frac{f(x)}{g(x)}=\lim_{x\to\pi/2}\frac{f'(x)}{g'(x)}=\lim_{x\to\pi/2}\frac{e^x(\cos x-\sin x)}{g'(x)}=\frac{-e^{\pi/2}}{2}.$$

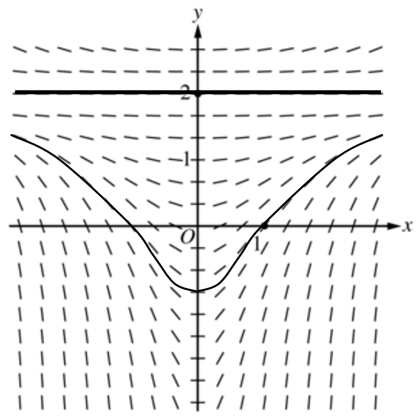
Notes:

1. Again, the simplest way to justify an extreme value of a continuous function on a closed interval is by the candidate test, illustrated here.

2. The recent emphasis on verifying that the conditions of a theorem are met when applying a mathematical result or theorem necessitates the separate statements about the limits of the numerator and denominator.

Question AB-6

(a)



(b) At the point $(1,0)$, $\frac{dy}{dx}=\frac{4}{3}$. The tangent line is $y=\frac{4}{3}(x-1)$. At $x=0.7$, $y=\frac{4}{3}(0.7-1)=-0.4$.

(c) $\int (y-2)^{-2}\,dy=\int \frac{x}{3}\,dx \Rightarrow \frac{-1}{y-2}=\frac{x^2}{6}+C$. Using the initial condition, $\frac{1}{2}=\frac{1}{6}+C \Rightarrow C=\frac{1}{3}$. So $\frac{-1}{y-2}=\frac{x^2}{6}+\frac{1}{3} \Rightarrow \frac{1}{y-2}=-\frac{x^2+2}{6} \Rightarrow y=2-\frac{6}{x^2+2}$.

2018 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See as AB Question 1.

Question BC-2

(a) $p'(25) \approx -1.179$ million cells per cubic meter per meter. At the depth of 25 meters, the density of plankton cells is decreasing at the rate of 1.179 millions of cells per cubic meter per meter.[1]

(b) $\int_0^{30} 3p(h)\,dh \approx 1675.415$. There are approximately 1675 million plankton cells in the column of water.

(c) The number of plankton cells in the entire column, in millions, is given $\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh$. Since $\int_{30}^{K} 3f(h)\,dh \le \int_{30}^{K} 3u(h)\,dh \le \int_{30}^{\infty} 3u(h)\,dh = 315$, $\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh \le 1675.415 + 315 = 1990.415$ million. This is less than 2000 million.

(d) The total distance traveled is $\int_0^1 \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt \approx 757.456$ meters.

Notes:

1. Use the same language as in the statement of the problem when interpreting the derivative of $p(h)$.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) $\frac{1}{2}\int_{\pi/3}^{5\pi/3} 16-\left(3+2\cos\theta\right)^2 d\theta$.

(b) $r\left(\frac{\pi}{2}\right)=3$; $\frac{dr}{d\theta}=-2\sin\theta \Rightarrow \left.\frac{dr}{d\theta}\right|_{\frac{\pi}{2}}=-2$.

$x=r\cos\theta \Rightarrow \frac{dx}{d\theta}=-r\sin\theta+\frac{dr}{d\theta}\cos\theta \Rightarrow \left.\frac{dx}{d\theta}\right|_{\frac{\pi}{2}}=-3$.

$y=r\sin\theta \Rightarrow \frac{dy}{d\theta}=r\cos\theta+\frac{dr}{d\theta}\sin\theta \Rightarrow \left.\frac{dy}{d\theta}\right|_{\frac{\pi}{2}}=-2$.

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2}=\left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\pi/2}=\frac{-2}{-3}=\frac{2}{3}$. The slope of the tangent line equation is $\frac{2}{3}$.

(c) $\frac{dr}{dt}=-2\sin\theta\frac{d\theta}{dt}=3 \Rightarrow \frac{d\theta}{dt}=\frac{3}{-2\sin\theta}$. When $\theta=\frac{\pi}{3}$, $\frac{d\theta}{dt}=\frac{3}{-2\sin\frac{\pi}{3}}=\frac{3}{-\sqrt{3}}=-\sqrt{3}$ radians per second. [1]

Notes:

1. You do not need to simplify $\frac{3}{-2\sin\frac{\pi}{3}}$.

Question BC-6

(a) The Maclaurin series for $\ln\left(1+\frac{x}{3}\right)$ is

$$\frac{x}{3}-\frac{\left(\frac{x}{3}\right)^2}{2}+\frac{\left(\frac{x}{3}\right)^3}{3}-\frac{\left(\frac{x}{3}\right)^4}{4}+\ldots+\frac{(-1)^{n+1}x^n}{3^n\cdot n}+\ldots=\frac{x}{3}-\frac{x^2}{3^2\cdot 2}+\frac{x^3}{3^3\cdot 3}-\frac{x^4}{3^4\cdot 4}+\ldots+\frac{(-1)^{n+1}x^n}{3^n\cdot n}+\ldots$$

The Maclaurin series for *f* is

$$\frac{x^2}{3}-\frac{x^3}{3^2\cdot 2}+\frac{x^4}{3^3\cdot 3}-\frac{x^5}{3^4\cdot 4}+\ldots+\frac{(-1)^{n+1}x^{n+1}}{3^n\cdot n}+\ldots$$

(b) Applying the ratio test, $\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\frac{x^{n+2}}{3^{n+1}\cdot(n+1)}}{\frac{x^{n+1}}{3^n\cdot n}}\right|=\left|\frac{x}{3}\cdot\frac{n}{n+1}\right| \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x}{3}\right|$. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

when $-3<x<3$. At $x = 3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}=3\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which is 3 times the alternating harmonic series, and converges. At $x = -3$, the series is $\sum_{n=1}^{\infty}\frac{3}{n}=3\sum_{n=1}^{\infty}\frac{1}{n}$ which is 3 times the harmonic series, and diverges. The interval of convergence is $-3<x\le 3$.

(c) The upper bound for $\left|P_4(2)-f(2)\right|$ obtained from the alternating series error bound is the absolute value of the next term, which is $\frac{2^5}{3^4\cdot 4}=\frac{32}{324}$. [1]

Notes:

1. Perhaps a more useful upper bound for the error is 0.1, since $\frac{32}{324}<0.1$.