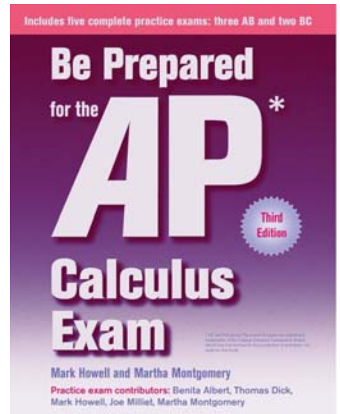

Third Edition

Be Prepared
for the

AP

Calculus
Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2017 AB
AP Calculus Free-Response Solutions and Notes

Question AB-1

(a) The left Riemann sum approximation is $50.3 \cdot 2 + 14.4 \cdot 3 + 6.5 \cdot 5$ cubic feet. [1]

(b) This is an overestimate of the volume of the tank since $A(h)$ is decreasing.

(c) The volume is $\int_0^{10} f(h)\,dh \approx 101.325$ cubic feet. [2]

(d) When the height of the water is h feet, the volume of the water $V = \int_0^h f(u)\,du$. The volume is changing at the rate of $\frac{dV}{dt} = f(h) \cdot \frac{dh}{dt}$. When $h = 5$,
$\frac{dV}{dt} = f(5) \cdot 0.26 \approx 1.694$ cubic feet per minute.

Notes:

1. You can leave the answer unsimplified. For the record, it is 176.3 cubic feet.

2. Use the given function names in your answers; do not repeat their formulas to avoid transcription errors.

Question AB-2

(a) $\int_0^2 f(t)\,dt \approx 20.051$ pounds.

(b) $f'(7) \approx -8.120$. [1] The rate at which customers remove bananas from the display table is decreasing at the rate of 8.120 pounds of bananas per hour per hour 7 hours after the store opened.

(c) $f(5) \approx 13.796$ and $g(5) \approx 11.533$. Since $f(5) > g(5)$, the bananas are being removed by customers faster than store employees add them at $t = 5$, so the number of bananas is decreasing at that time.

(d) $50 - \int_0^8 f(t)\,dt + \int_3^8 g(t)\,dt \approx 23.347$. [2]

Notes:

1. Use your calculator — no need for symbolic differentiation.

2. The fact that *f* and *g* are not defined at the lower limits of integration does not affect the values of the two integrals.

Question AB-3

(a) $f(x) = 7 + \int_{-2}^{x} f'(t)\,dt$, so $f(-6) = 7 + \int_{-2}^{-6} f'(t)\,dt = 7 - \frac{1}{2} \cdot 4 \cdot 2 = 3$, and

$f(5) = 7 + \int_{-2}^{5} f'(t)\,dt = 7 - \frac{1}{2} \cdot \pi \cdot 2^2 + \frac{1}{2} \cdot 3 \cdot 2$. [1]

(b) f is increasing on the intervals $-6 \le x \le -2$ and $2 \le x \le 5$ since f is continuous on $[-6, 5]$ and $f'(x) > 0$ on $(-6, -2)$ and $(2, 5)$.

(c) The only points where f could have an absolute minimum are $x = -6, x = -2, x = 2$, and $x = 5$. $f(-6) = 3$, $f(-2) = 7$, $f(2) = 7 - 2\pi$, $f(5) = 10 - 2\pi$. The absolute minimum is $7 - 2\pi$. [2]

(d) $f''(-5) = -\frac{1}{2}$; $f''(3)$ does not exist, because $\lim_{x \to 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2$ and

$\lim_{x \to 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$.

Notes:

1. You can leave the answer unsimplified. For the record, it is $10 - 2\pi$.

2. The candidate test is the easiest way to justify absolute extrema for a continuous function on a closed interval.

Question AB-4

(a) $\left.\dfrac{dH}{dt}\right|_{t=0} = -\dfrac{1}{4}(91-27) = -16$. [1] An equation of the tangent line is $y - 91 = -16t$.
At $t = 3$, $y = 91 - 48 = 43$. The temperature of the potato is approximately 43 °C.

(b) $\dfrac{d^2H}{dt^2} = \dfrac{d}{dt}\left(-\dfrac{1}{4}(H-27)\right) = -\dfrac{1}{4}\cdot\dfrac{dH}{dt} = \dfrac{1}{16}(H-27)$. Since $H > 27$ for all $t > 0$, $\dfrac{d^2H}{dt^2} > 0$ for all $t > 0$. Therefore, the tangent line approximation is an underestimate of the temperature.

(c) Separating variables, we get $(G-27)^{-2/3}\,dG = -dt$.
$\int (G-27)^{-2/3}\,dG = \int -dt \;\Rightarrow\; 3\cdot(G-27)^{1/3} = -t + C$. From the initial condition $G(0) = 91$, $C = 3\cdot(91-27)^{1/3} = 12$. So $3\cdot(G-27)^{1/3} = -t + 12$
$\Rightarrow\; (G-27)^{1/3} = -\dfrac{t}{3} + 4 \;\Rightarrow\; G = 27 + \left(-\dfrac{t}{3}+4\right)^3$. At $t = 3$, $G = 27 + 27 = 54$ °C.

Notes:

1. $H(t)$ models the temperature for $t \ge 0$, but we can assume that $\dfrac{dH}{dt}$ is defined on a larger interval including $t = 0$.

Question AB-5

(a) Particle P is moving to the left when $x_P'(t)=\dfrac{2t-2}{t^2-2t+10}<0 \;\Rightarrow$
$2t-2<0 \;\Rightarrow\; 0\le t<1$.

(b) $v_Q(t)=(t-3)(t-5)$ so $v_Q(t)<0$ for $3<t<5$. The particles never move left at the same time, but for $1<t<3$ and $5<t\le 8$, both particles are moving to the right.

(c) $a_Q(t)=v_Q'(t)=2t-8 \;\Rightarrow\; a_Q(2)=-4<0$. $v_Q(2)=(-1)\cdot(-3)=3>0$. $a_Q(2)$ and $v_Q(2)$ have opposite signs, so the speed of the particle is decreasing at $t=2$.

(d) Particle Q first changes direction at $t=3$. Its position at that time is
$$5+\int_0^3\left(t^2-8t+15\right)dt=5+\left(\frac{t^3}{3}-4t^2+15t\right)\Bigg|_0^3=5+9-36+45.$$ [1]

Notes:

1. You can leave the answer unsimplified. For the record, it is 23.

Question AB-6

(a) $f'(x) = -2\sin(2x) + e^{\sin(x)} \cdot \cos(x) \Rightarrow f'(\pi) = -1$. The slope of the tangent line is -1.

(b) $k'(x) = h'(f(x)) \cdot f'(x) \;\Rightarrow\; k'(\pi) = h'(f(\pi)) \cdot f'(\pi)$.

$f(\pi) = \cos(2\pi) + e^{\sin \pi} = 2$; $h'(2) = -\frac{1}{3}$; $f'(\pi) = -1 \;\Rightarrow\; k'(\pi) = -\frac{1}{3} \cdot (-1) = \frac{1}{3}$.

(c) $m'(x) = g(-2x) \cdot h'(x) - 2g'(-2x) \cdot h(x)$

$m'(2) = g(-4) \cdot h'(2) - 2g'(-4) \cdot h(2) = 5 \cdot \left(-\frac{1}{3}\right) - 2 \cdot (-1) \cdot \left(-\frac{2}{3}\right) = -3$. [1]

(d) $\frac{g(-3) - g(-5)}{-3 - (-5)} = \frac{-8}{2} = -4$. Since g is differentiable, it is also continuous. So by the Mean Value Theorem applied to g on $[-5, -3]$, there must be a c in $(-5, -3)$ such that $g'(c) = -4$.

Notes:

1. You can leave the answer unsimplified.

2017 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) Area is $\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(f(\theta)\right)^2 d\theta \approx 0.648$. [1]

(b) $\frac{1}{2}\int_0^{k}\left(\left(g(\theta)\right)^2-\left(f(\theta)\right)^2\right)d\theta=\frac{1}{2}\int_k^{\frac{\pi}{2}}\left(\left(g(\theta)\right)^2-\left(f(\theta)\right)^2\right)d\theta$.

(c) $w(\theta)=g(\theta)-f(\theta)$. The average distance $w_A=\frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}w(\theta)\,d\theta \approx 0.485$.

(d) $w(\theta)\approx 0.485 \;\Rightarrow\; \theta\approx 0.518$. $w'(0.518)\approx -0.582$. [2] Since $w'(0.518)<0$, $w(\theta)$ is decreasing.

Notes:

1. Use the given function names in the setups for parts (a), (b), and (c). Doing so saves time and is less prone to transcription errors.

2. Use your calculator to calculate the derivative — no need for symbolic differentiation.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) $f'(x) = -\dfrac{3(4x-7)}{(2x^2-7x+5)^2} \;\Rightarrow\; f'(3) = -\dfrac{15}{(18-21+5)^2}$. ☐1

(b) $f'(x) = 0 \;\Rightarrow\; x = \dfrac{7}{4}$. Since $f'(x)$ changes sign from positive to negative at $x = \dfrac{7}{4}$, f has a relative maximum at $x = \dfrac{7}{4}$.

(c) $\int_5^\infty f(x)\,dx = \lim\limits_{b\to\infty}\int_5^b \left(\dfrac{2}{2x-5} - \dfrac{1}{x-1}\right)dx = \lim\limits_{b\to\infty}\left(\ln(2x-5) - \ln(x-1)\right)\Big|_1^b$

$= \lim\limits_{b\to\infty}\left(\ln\left(\dfrac{2x-5}{x-1}\right)\right)\Bigg|_5^b = \lim\limits_{b\to\infty}\left(\ln\left(\dfrac{2b-5}{b-1}\right) - \ln\left(\dfrac{5}{4}\right)\right) = \ln(2) - \ln\left(\dfrac{5}{4}\right)$. ☐2

(d) Since $f(x)$ is continuous, positive, and decreasing for $x > 5$, and $\int_5^\infty f(x)\,dx$ converges, the series converges by the Integral Test.

☐ **Notes:**

1. You can leave the answer unsimplified. For the record, it is $-\dfrac{15}{4}$.

2. $= \ln\left(\dfrac{8}{5}\right)$.

Question BC-6

(a) $f''(0) = -1 \cdot f'(0) = -1$; $f'''(0) = -2 \cdot f''(0) = 2$; $f^{(4)}(0) = -3 \cdot f'''(0) = -6$. So the first four nonzero terms for the Maclaurin series for f are $x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. The general term is $(-1)^{n+1}\frac{x^n}{n}$.

(b) At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ which is the alternating harmonic series. It converges by the Alternating Series Test. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the harmonic series, which diverges. Therefore, the series converges conditionally at $x = 1$.

(c) $g(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \ldots + (-1)^{n+1}\frac{x^{n+1}}{n(n+1)} + \ldots$.

(d) $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right|$ is bounded by the magnitude of the first omitted term, which is $\frac{\left(\frac{1}{2}\right)^5}{20} = \frac{1}{640} < \frac{1}{500}$.