Be Prepared

for the

AP

Calculus Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2015 AB
AP Calculus Free-Response Solutions and Notes

Question AB-1

(a) $\int_0^8 R(t)\,dt \approx 76.570$. [1]

(b) $R(3) \approx 5.086$ and $D(3) = 5.4$. Since $R(3) < D(3)$, the amount of water is decreasing.

(c) The amount *A* of water in the drain at time *t*, for $0 \le t \le 8$, is given by $A(t) = 30 + \int_0^t \left(R(u) - D(u)\right) du$, so $A'(t) = R(t) - D(t)$. $A'(t) = 0$ at $t \approx 3.2716584$. Let $T = 3.2716584$. [2]

The candidates for the minimum are *t* = 0, *t* = *T*, and *t* = 8. $A(0) = 30$, $A(T) = 30 + \int_0^T \left(R(u) - D(u)\right) du \approx 27.965$, and $A(8) \approx 48.544$, so the minimum occurs at $t = T \approx 3.272$ hours.

(d) $50 = 48.544 + \int_8^w \left(R(t) - D(t)\right) dt$. [3]

Notes:

1. Use the given names for the functions in your answers, rather than their formulas, to avoid transcription errors.

2. Store this value in your calculator.

3. Or, since the pipe does not overflow on the interval $0 \le t \le 8$, the equation $50 = 30 + \int_0^w \left(R(t) - D(t)\right) dt$ is also correct.

Question AB-2

(a) Solving $f(x) = g(x)$ gives $x \approx 1.0328319$. Let $A = 1.0328319$. [1]

$\int_0^A (g(x) - f(x))\,dx + \int_A^2 (f(x) - g(x))\,dx \approx 0.9974 + 1.0069 \approx 2.004$. [2]

(b) $\int_A^2 (f(x) - g(x))^2\,dx \approx 1.283$.

(c) $\frac{d}{dx}(f(x) - g(x))\Big|_{x=1.8} = f'(1.8) - g'(1.8) \approx -3.812$. [3]

Notes:

1. Store this value in your calculator and use it in subsequent calculations. Keep high precision.

2. A quicker and simpler solution for Part (a) is $\int_0^2 |g(x) - f(x)|\,dx \approx 2.004$.

3. Or –3.811, if the value is truncated at the third decimal place.

Question AB-3

(a) $v'(16) \approx \dfrac{240-200}{20-12}\ \dfrac{\text{meters/minute}}{\text{minute}}$. [1]

(b) $\int_0^{40} |v(t)|\,dt$ is the total distance in meters that Johanna jogged during the time interval from 0 to 40 minutes. [2] The right Riemann sum approximation is $200 \cdot 12 + 240 \cdot 8 + 220 \cdot 4 + 150 \cdot 16$ meters. [3]

(c) Bob's acceleration at $t = 5$ is $B'(5) = 3 \cdot 5^2 - 12 \cdot 5\ \dfrac{\text{meters/minute}}{\text{minute}}$. [4]

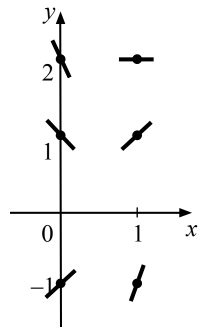
(d) $\dfrac{1}{10}\int_0^{10} B(t)\,dt = \dfrac{1}{10}\left(\dfrac{t^4}{4} - 2t^3 + 300t\right)\Bigg|_0^{10} = \dfrac{1}{10}(2500 - 2000 + 3000)\ \dfrac{\text{meters}}{\text{minute}}$. [5]

Notes:

1. No need to simplify — saves time and helps avoid arithmetic errors.
2. It is important to address both the dependent and independent variables with correct units, distance in meters and time in minutes.
3. =7600 meters. No need to simplify.
4. =15. No need to simplify.
5. =350. No need to simplify.

Question AB-4

(a)



(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II, $x < 0$ and $y > 0$, so $\frac{d^2y}{dx^2} > 0$. Therefore, the solution curves are concave up in Quadrant II.

(c) At $(2, 3)$, $\frac{dy}{dx} = 4 - 3 = 1 \neq 0$. Therefore, the *f* has neither a relative maximum nor a relative minimum at $x = 2$.

(d) If $y = mx + b$ is a solution, then $\frac{d^2y}{dx^2} = 2 - 2x + y = 0$, so $y = 2x - 2 \Rightarrow m = 2$ and $b = -2$. [1]

Notes:

1. Alternative solution: if $y = mx + b$ is a solution, then $\frac{dy}{dx} = m \Rightarrow m = 2x - (mx + b) = (2 - m)x - b$. This equation must be true for all *x*, so $2 - m = 0$ and $m = -b \Rightarrow m = 2,\ b = -2$.

Question AB-5

(a) *f* has a relative maximum at $x = -2$, because $f'(x)$ changes sign from positive to negative there.

(b) For the graph of *f* to be concave down, f' must be decreasing. For *f* to be decreasing, f' must be negative. These two conditions together are met on the intervals $-2 < x < -1$ and $1 < x < 3$.

(c) The graph of *f* has points of inflection at $x = -1$, $x = 1$, and $x = 3$ since f' has relative extrema at these points. [1]

(d) $f(x) = 3 + \int_1^x f'(t)\,dt$. $f(4) = 3 + \int_1^4 f'(t)\,dt = 3 + (-12) = -9$ and

$f(-2) = 3 + \int_1^{-2} f'(t)\,dt = 3 - (-9) = 12$.

Notes:

1. "Since f'' changes sign at these points" is also an acceptable justification.

Question AB-6

(a) At (–1, 1), $\frac{dy}{dx}=\frac{1}{4}$. An equation for the tangent line is $y-1=\frac{1}{4}(x+1)$.

(b) A vertical tangent line requires $3y^2=x$ and $y\neq 0$. Substituting $3y^2$ for x in the equation of the curve, we get $y^3-\left(3y^2\right)y=2\Rightarrow -2y^3=2 \quad\Rightarrow\quad y=-1$. When $y=-1$, $x=3$. So the point on the curve with a vertical tangent is $(3,-1)$.

(c) $\frac{d^2y}{dx^2}=\frac{\left(3y^2-x\right)\frac{dy}{dx}-y\left(6y\frac{dy}{dx}-1\right)}{\left(3y^2-x\right)^2}$. 1 At (–1, 1), this is $\frac{4\cdot\frac{1}{4}-1\left(6\cdot\frac{1}{4}-1\right)}{4^2}$. 2

Notes:

1. Don't bother trying to simplify this expression: just substitute.

2. $=\frac{1}{32}$. Do not simplify.

2015 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) $x(2) = x(1) + \int_1^2 x'(t)\,dt = 3 + \int_1^2 \cos\left(t^2\right) dt \approx 2.557$. [1]

(b) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2 \;\Rightarrow\; \dfrac{e^{0.5t}}{\cos\left(t^2\right)} = 2 \;\Rightarrow\; t \approx 0.840$.

(c) Speed = $\sqrt{\left(\cos\left(t^2\right)\right)^2 + \left(e^{0.5t}\right)^2} = 3 \;\Rightarrow\; t \approx 2.196$. [2]

(d) Distance traveled = $\int_0^1 \sqrt{\left(\cos\left(t^2\right)\right)^2 + \left(e^{0.5t}\right)^2}\,dt \approx 1.595$. [3]

Notes:

1. Or 2.556.

2. Or 2.195.

3. Or 1.594.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) When $k = 3$, $f(4) = \dfrac{1}{16-12} = \dfrac{1}{4}$ and $f'(4) = \dfrac{3 - 2\cdot 4}{\left(4^2 - 3\cdot 4\right)^2} = -\dfrac{5}{16}$. The tangent line is $y - \dfrac{1}{4} = -\dfrac{5}{16}(x-4)$.

(b) When $k = 4$, $f'(x) = \dfrac{4-2x}{\left(x^2-4x\right)^2}$. $f'(2) = 0$ and $f'(x)$ changes sign from positive to negative at $x = 2$, so f has a local maximum at $x = 2$.

(c) $f'(-5) = \dfrac{k+10}{(25+5k)^2} = 0 \;\Rightarrow\; k = -10$. [1,2]

(d) $\dfrac{1}{x^2-6x} = \dfrac{1}{x(x-6)} = \dfrac{A}{x} + \dfrac{B}{x-6} \Rightarrow 1 = A(x-6) + Bx$. Let $x = 6$, so $B = \dfrac{1}{6}$.

Let $x = 0$, so $A = -\dfrac{1}{6}$. Therefore, $\dfrac{1}{x^2-6x} = \dfrac{-1/6}{x} + \dfrac{1/6}{x-6}$.

$$\int \frac{1}{x^2-6x}\,dx = \int\left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right)dx = -\frac{1}{6}\ln|x| + \frac{1}{6}\ln|x-6| + C.$$

Notes:

1. To complete the solution, we should verify that $f(x)$ is defined at $x = -5$ when $k = -10$, that is, $x^2 - kx$ is not 0 at $x = -5$. It seems unlikely that this will be required to get full credit; we will know for sure when the question is graded.

2. $f'(-5)$ is undefined when $k = -5$, but $x = -5$ is not a critical point of f because $f(-5)$ is also undefined.

Question BC-6

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^n}{n+1} \cdot x^{n+1} \cdot \frac{n}{3^{n-1}} \cdot \frac{1}{x^n}\right| = \left|\frac{n}{n+1} \cdot 3x\right|$.

$\lim_{n\to\infty}\left|\frac{n}{n+1} \cdot 3x\right| = |3x| < 1 \Rightarrow |x| < \frac{1}{3}$, so $R = \frac{1}{3}$.

(b) The series for f' is $1 - 3x + 9x^2 - 27x^3 + \ldots$. This is a geometric series with the first term 1 and common ratio $-3x$, so $f'(x) = \frac{1}{1+3x}$.

(c) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$. So $e^x f(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \ldots\right)$. The third degree Taylor polynomial for $g(x) = e^x f(x)$ can be found by using the distributive property (and only keeping the terms with powers ≤ 3):

$P_3(x) = x - \frac{3}{2}x^2 + 3x^3 + x^2 - \frac{3}{2}x^3 + \frac{x^3}{2} = x + \left(-\frac{3}{2} + 1\right)x^2 + \left(3 - \frac{3}{2} + \frac{1}{2}\right)x^3$. [1]

Notes:

1. Leave it at this to save time and avoid arithmetic mistakes. (This is $x - \frac{1}{2}x^2 + 2x^3$.)