Be Prepared
for the

AP

Calculus Exam

Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

Chapter 10.   Annotated Solutions to Past Free-Response Questions

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

2014 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) The average rate of change is $\frac{A(30)-A(0)}{30-0} \approx -0.197$ pounds per day.

(b) $A'(15) \approx -0.164$ pounds per day. On day 15, the amount of grass clippings remaining in the bin is decreasing at the rate of 0.164 pounds per day.

(c) The average amount of grass clippings is $\frac{1}{30}\int_0^{30} A(t)\,dt \approx 2.752635$ pounds.[1]
Solving $A(t) = 2.752635$ gives $t \approx 12.415$ days.

(d) $A(30) \approx 0.782928$ and $A'(30) \approx -0.055976$.[2]
$L(t) = A(30) + A'(30)\cdot(t-30)$. Solving $L(t) = 0.5$ gives $t \approx 35.054$ days.

Notes:

1. Store this result in a calculator variable, then use the stored value to solve the equation.

2. Again, store these results in calculator variables (keep greater accuracy to assure that the final answer is correct), then use the stored values in the formula for $L(t)$.

Question AB-2

(a) Solving $f(x) = 4$ 🖩 gives $x = 0$ and $x = 2.3$. [1]

$V = \pi \int_0^{2.3} (4+2)^2 - (f(x)+2)^2 \, dx$ 🖩 ≈ 98.868.

(b) The area of each isosceles right triangle is $\frac{1}{2} \cdot (4 - f(x))^2$. The volume of the solid is $\int_0^{2.3} \frac{1}{2} \cdot (4 - f(x))^2 \, dx$ 🖩 ≈ 3.574.

(c) $\int_0^k (4 - f(x))\, dx = \int_k^{2.3} (4 - f(x))\, dx$. [2]

Notes:

1. This is the exact value for the *x*-coordinate of the point. In this case, it's just as easy to enter 2.3 in subsequent calculations.

2. Or: $\int_0^k (4 - f(x))\, dx = \frac{1}{2} \int_0^{2.3} (4 - f(x))\, dx$.

Question AB-3

(a) Using the areas of two triangles, $g(3)=\int_{-3}^{3} f(t)\,dt=\frac{1}{2}\cdot 5\cdot 4-\frac{1}{2}\cdot 1\cdot 2$ [1] $=10-1=9$. [2]

(b) The graph of g is increasing where $g'(x)=f(x)\geq 0$. The graph of g is concave down where $g''(x)=f'(x)<0$. Both of these conditions hold for $-5<x<-3$ and for $0<x<2$.

(c) $h'(x)=\frac{5x\cdot g'(x)-5\cdot g(x)}{25x^2} \Rightarrow h'(3)=\frac{15\cdot g'(3)-5\cdot g(3)}{25\cdot 9}=\frac{15\cdot f(3)-5\cdot g(3)}{25\cdot 9}=$ $\frac{15\cdot(-2)-5\cdot 9}{9\cdot 25}$ [1] $=-\frac{1}{3}$.

(d) $p'(x)=f'(x^2-x)\cdot(2x-1)$. The slope at $x=-1$ is equal to $p'(-1)=f'(2)\cdot(-3)=(-2)\cdot(-3)=6$.

Notes:

1. You can leave it at that to avoid arithmetic mistakes.

2. Or, using the areas of three triangles, $g(3)=\frac{1}{2}\cdot 3\cdot 4+\frac{1}{2}\cdot 2\cdot 4-\frac{1}{2}\cdot 1\cdot 2=9$.

Question AB-4

(a) The average acceleration is the average rate of change of velocity, which is $\frac{v(8)-v(2)}{8-2}=\frac{-120-100}{6}=-\frac{220}{6}$ meters per minute per minute.

(b) Since $v_A(t)$ is differentiable, it is also continuous. Since -100 is between $v_A(5)=40$ and $v_A(8)=-120$, the Intermediate Value Theorem applied to $v_A(t)$ on the interval [5, 8] guarantees that $v_A(t)=-100$ for some t between 5 and 8.

(c) The position at $t = 12$ is $300+\int_2^{12} v_A(t)\,dt$. Using a trapezoidal sum approximation, this is approximately
$300+(5-2)\frac{100+40}{2}+(8-5)\frac{40+(-120)}{2}+(12-8)\frac{(-120)+(-150)}{2}$ 1 meters.

(d) Let $x(t)$ and $y(t)$ be the positions of trains A and B, respectively, and $z(t)$ be the distance between the trains. Then $z^2=x^2+y^2 \;\Rightarrow\; 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt} \;\Rightarrow$
$z\frac{dz}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$. At $t = 2$, $x(t)=300$ and $y(t)=400$, so $z(t)=500$.
$\left.\frac{dx}{dt}\right|_{t=2}=v_A(2)=100$. $\frac{dy}{dt}=v_B(t)=-5t^2+60t+25$, so
$\left.\frac{dy}{dt}\right|_{t=2}=-5\cdot 2^2+60\cdot 2+25=125$. Therefore, at $t = 2$,
$500\frac{dz}{dt}=300\cdot 100+400\cdot 125 \;\Rightarrow\; \left.\frac{dz}{dt}\right|_{t=2}=\frac{300\cdot 100+400\cdot 125}{500}$ 2 meters per minute.

Notes:

1. $=-150$, so the train is approximately 150 meters west of Origin Station.

2. $=160$

Question AB-5

(a) f has only one relative minimum on $[-2, 3]$, at $x = 1$, because this is the only number on $[-2, 3]$ where $f'(x)$ changes sign from negative to positive.

(b) Since f is a twice-differentiable function, the Mean Value Theorem applies to $f'(x)$ on the interval $[-1, 1]$. Thus, there is a c in the open interval $(-1, 1)$ such that $f''(c) = \dfrac{f'(1) - f'(-1)}{2} = 0$. [1]

(c) $h'(x) = \dfrac{f'(x)}{f(x)} \Rightarrow h'(3) = \dfrac{f'(3)}{f(3)} = \dfrac{1/2}{7} = \dfrac{1}{14}$.

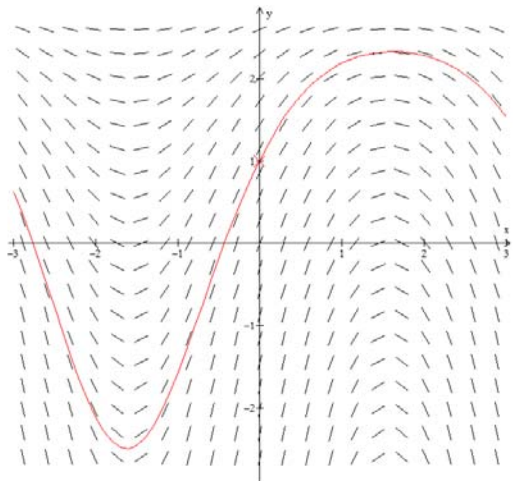
(d) An antiderivative for $f'(g(x))g'(x)$ is $f(g(x))$. So
$\int_{-2}^{3} f'(g(x))g'(x)\,dx = f(g(x))\Big|_{-2}^{3} = f(g(3)) - f(g(-2)) =$
$f(1) - f(-1) = 2 - 8 = -6$.

Notes:

1. Alternatively, since $f'(-1) = f'(1) = 0$, you can refer to Rolle's Theorem.

Question AB-6

(a)



(b) At (0, 1), the slope is $(3-1)\cos(0)=2$. An equation of the tangent line is $y-1=2x$. For x = 0.2, this gives y = 1.4. [1]

(c) Separating variables, we get $\frac{dy}{3-y}=\cos(x)\,dx \Rightarrow$ $\int\frac{dy}{3-y}=\int\cos(x)\,dx \Rightarrow -\ln|3-y|=\sin(x)+C$. Substituting the initial condition $f(0)=1$, we get $-\ln(2)=C \Rightarrow C=-\ln(2) \Rightarrow$ $-\ln|3-y|=\sin(x)-\ln(2) \Rightarrow \ln|3-y|=-\sin(x)+\ln(2) \Rightarrow$ $|3-y|=e^{-\sin(x)+\ln(2)}=2e^{-\sin(x)}$. Two possible solutions satisfy this equation, $y=3-2e^{-\sin(x)}$ and $y=3+2e^{-\sin(x)}$, but only the first one of them satisfies the initial condition $y(0)=1$. Therefore, $f(x)=3-2e^{-\sin(x)}$. [2]

Notes:

1. The statement $f(0.2)=1.4$ is incorrect and could result in lost points.

2. The domain of this solution is all real numbers.

2014 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) The curves intersect at $\theta = \frac{\pi}{2}$ and $\theta = \pi$. The area of R consists of the area in the first quadrant plus the area of the quarter circle in the second quadrant:
Area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(3 - 2\sin(2\theta)\right)^2 d\theta + \frac{9\pi}{4} \approx 9.708$.

(b) $x = r\cos\theta \;\Rightarrow\; \frac{dx}{d\theta} = \frac{d}{d\theta}\left(\left(3 - 2\sin(2\theta)\cdot\cos(\theta)\right)\right)$. Therefore,
$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{6}} \approx -2.366$.

(c) The distance between the curves as a function of θ is the difference between their respective values of r for a given θ, that is, $3 - \left(3 - 2\sin(2\theta)\right)$. The rate of change of that distance is $\frac{d}{d\theta}\left(3 - \left(3 - 2\sin(2\theta)\right)\right)$. At $\theta = \frac{\pi}{3}$, the rate of change is -2.[1]

(d) $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$. At $\theta = \frac{\pi}{6}$, this is $-2\cdot 3 = -6$.

Notes:

1. Or write $\frac{d}{d\theta}\left(3 - \left(3 - 2\sin(2\theta)\right)\right) = 2\frac{d}{d\theta}\sin(2\theta) = 4\cos(2\theta)$, which is equal to -2 when $\theta = \frac{\pi}{3}$.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) Area = $\int_0^1 \left(xe^{x^2} - (-2x)\right)dx = \left(\frac{1}{2}e^{x^2} + x^2\right)\Big|_0^1 = \frac{1}{2}e + 1 - \frac{1}{2}$. ▢1

(b) Volume = $\pi\int_0^1 \left(\left(xe^{x^2} + 2\right)^2 - \left(-2x+2\right)^2\right)dx$.

(c) Given $y = xe^{x^2}$, $\frac{dy}{dx} = 2x^2e^{x^2} + e^{x^2}$. The vertical line $x = 1$ intersects $y = xe^{x^2}$ at $y = e$ and intersects $y = -2x$ at $y = -2$, so the length of the vertical segment is $e + 2$. The length of the linear segment with slope –2 can be evaluated using the Pythagorean Theorem (or the distance formula): length = $\sqrt{1^2 + 2^2} = \sqrt{5}$.

Perimeter = $e + 2 + \sqrt{5} + \int_0^1 \sqrt{1 + \left(2x^2e^{x^2} + e^{x^2}\right)^2}\,dx$.

▢ **Notes:**

1. $= \frac{1}{2}e + \frac{1}{2}$.

Question BC-6

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-1)^{n+1}}{n+1} \cdot \frac{n}{2^n (x-1)^n}\right| = \left|\frac{2n}{n+1} \cdot (x-1)\right|$.

$\lim_{n\to\infty}\left|\frac{2n}{n+1} \cdot (x-1)\right| = 2|x-1| < 1 \;\Rightarrow\; |x-1| < \frac{1}{2} \;\Rightarrow\; R = \frac{1}{2}$.

(b) The series for *f* is $2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 - ... + (-1)^{n+1}\frac{2^n}{n}(x-1)^n + ...$. The series for f' is $2 - 4(x-1) + 8(x-1)^2 - ... + (-1)^{n+1} 2^n (x-1)^{n-1} + ...$.

(c) The first term of the series for f' is 2 and the common ratio is $-2(x-1)$, so the series converges to $f'(x) = \frac{2}{1+2(x-1)} = \frac{2}{2x-1} \;\Rightarrow$

$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$. $f(1) = 0 \;\Rightarrow\; C = 0 \;\Rightarrow\; f(x) = \ln|2x-1|$.

Since $\frac{1}{2} < x < \frac{3}{2}$, $2x-1$ is positive, so $f(x) = \ln(2x-1)$.