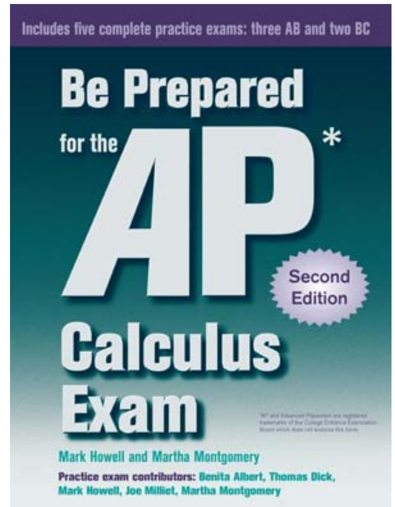

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AP

Calculus Exam

Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

Chapter 10. Annotated Solutions to Past Free-Response Questions

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

2012 AB
AP Calculus Free-Response Solutions and Notes

Question AB-1

(a) $W'(12) \approx \dfrac{W(15)-W(9)}{15-9} = \dfrac{67.9-61.8}{15-9} \approx 1.017$ degrees/minute.[1] The average rate of change of the temperature of the water in degrees Fahrenheit per minute from $t = 9$ to $t = 15$ minutes is 1.017.[1] This approximates $W'(12)$, the instantaneous rate of change of the temperature at $t = 12$ minutes, indicating that the temperature is increasing at approximately 1.017 degrees Fahrenheit per minute at that time.

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16.0$ degrees.[1] This is the net change in temperature of the water in degrees Fahrenheit from 0 to 20 minutes.

(c) $\dfrac{1}{20}\int_0^{20} W(t)\,dt \approx \dfrac{1}{20}\left(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9\right) = 60.79$. This value underestimates the average temperature of the water, because a left Riemann sum for an integral of an increasing function underestimates the integral.

(d) $W(25) = W(20) + \int_{20}^{25} W'(t)\,dt \approx 71.0 + 2.043 = 73.043$ degrees Fahrenheit.

Notes:

1. It is important to mention the units both for the variable t (minutes) and the temperature (degrees Fahrenheit).

Question AB-2

(a) $\ln x = 5 - x$ at $x = 3.693441$. Let $A = 3.693441$.[1]

Area = $\int_1^A \ln x\,dx + \int_A^5 (5-x)\,dx \approx 2.9858 \approx 2.986$.

(b) Volume = $\int_1^A (\ln x)^2\,dx + \int_A^5 (5-x)^2\,dx$.

(c) $\ln x = k$ at $x = e^k$ and $5 - x = k$ at $x = 5 - k$.

$\int_{e^k}^A (\ln x - k)\,dx + \int_A^{5-k} (5 - x - k)\,dx = \frac{1}{2}\left(\int_1^A \ln x\,dx + \int_A^5 (5-x)\,dx\right)$.[2]

Notes:

1. Store this number in your calculator, and use the stored value in the subsequent calculations.
2. Or: $\int_{e^k}^A (\ln x - k)\,dx + \int_A^{5-k} (5 - x - k)\,dx = \frac{1}{2} \cdot 2.9858$.

   A simpler equation results from integrating with respect to *y*:
   $\int_0^k (5 - y - e^y)\,dy = \frac{1}{2}\int_0^B (5 - y - e^y)\,dy$ or $\int_0^k (5 - y - e^y)\,dy = \int_k^B (5 - y - e^y)\,dy$,
   where $B = 5 - A$.

Question AB-3

(a) $g(2)=\int_1^2 f(t)\,dt=-\frac{1}{4}$ (negative area of a right triangle with legs 1 and $f(2)$);

$g(-2)=\int_1^{-2} f(t)\,dt=-\int_{-2}^{1} f(t)\,dt=-\left(\frac{3}{2}-\frac{\pi}{2}\right)$. [1]

(b) $g'(-3)=f(-3)=2$; $g''(-3)=f'(-3)=1$.

(c) The graph of g has horizontal tangents where $g'(x)=f(x)=0$, namely at $x=-1$ and $x=1$. $g(x)$ has a relative maximum at $x=-1$ because $g'(x)$ changes sign from positive to negative there. $g(x)$ has neither a relative maximum nor a relative minimum at $x=1$ because $g'(x)$ does not change sign there.

(d) The graph of g has three points of inflection, at $x=-2$, $x=0$, and $x=1$, because $g'(x)$ changes from increasing to decreasing at $x=-2$ and at $x=1$, and from decreasing to increasing at $x=0$. [2]

Notes:

1. When evaluating definite integrals of functions defined by geometric figures, it might be easier to work from left to right.
2. An alternative justification involves relating the sign change in $g''(x)$ from positive to negative at $x=-2$ and at $x=1$ and from negative to positive at $x=0$.

Question AB-4

(a) $f'(x) = \frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25-x^2}}$.

(b) $f(-3) = \sqrt{25-9} = 4; f'(-3) = \frac{3}{4}$. The tangent line is $y-4 = \frac{3}{4}(x+3)$.

(c) By definition, g is continuous at $x=-3$ if $\lim\limits_{x\to-3} g(x) = g(-3)$.

$\lim\limits_{x\to-3^-} g(x) = \lim\limits_{x\to-3^-} f(x) = f(-3) = 4$ and $\lim\limits_{x\to-3^+} g(x) = \lim\limits_{x\to-3^+} (x+7) = 4$. Therefore, $\lim\limits_{x\to-3} g(x)$ exists and is equal to $4 = f(-3) = g(-3)$. So g is continuous at $x=-3$.

(d) $\int_0^5 x\left(25-x^2\right)^{1/2} dx = -\frac{1}{2}\int_0^5 \left(25-x^2\right)^{1/2}(-2x)\,dx =$ [1]

$$-\frac{1}{2}\cdot\frac{2}{3}\left(25-x^2\right)^{\frac{3}{2}}\Bigg|_0^5 = -\frac{1}{3}\cdot(0-125) = \frac{125}{3}.$$

Notes:

1. Or use more formal u-substitution $u = 25-x^2$, $\frac{du}{dx} = -2x$, $-\frac{1}{2}du = x\,dx$.

Question AB-5

(a) When $B = 40$, $\frac{dB}{dt} = \frac{1}{5} \cdot 60 = 12$, and when $B = 70$, $\frac{dB}{dt} = \frac{1}{5} \cdot 30 = 6$. Therefore, the bird is gaining weight faster when it weighs 40 grams than when it weighs 70 grams.

(b) $\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{1}{5}(100 - B)\right) = -\frac{1}{5} \cdot \left(\frac{dB}{dt}\right) = -\frac{1}{5} \cdot \left(\frac{1}{5}(100 - B)\right) = -\frac{1}{25} \cdot (100 - B)$. For $0 < B < 100$, $\frac{d^2B}{dt^2} < 0$, so the graph of B must be concave down, but the graph in the picture is not.

(c) (c) $\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt \Rightarrow -\ln|100 - B| = \frac{1}{5}t + C$. Substituting $(0, 20)$ we get $-\ln(80) = C \Rightarrow -\ln|100 - B| = \frac{1}{5}t - \ln(80) \Rightarrow \ln|100 - B| = \ln(80) - \frac{1}{5}t \Rightarrow$ $e^{\ln|100 - B|} = e^{\ln(80) - \frac{1}{5}t} \Rightarrow 100 - B = 80e^{-\frac{t}{5}} \Rightarrow$ [1] $B = 100 - 80e^{-\frac{t}{5}}$.

Notes:

1. Since $100 - B > 0$, $|100 - B| = 100 - B$.

Question AB-6

(a) The particle is moving to the left when $v(t) < 0$. This takes place when

$$\cos\left(\frac{\pi}{6}t\right) < 0 \iff \frac{\pi}{2} < \frac{\pi}{6}t < \frac{3\pi}{2} \iff 3 < t < 9.$$

(b) The total distance traveled is equal to $\int_0^6 |v(t)|\,dt$.

(c) $a(t) = v'(t) = -\frac{\pi}{6}\sin\left(\frac{\pi t}{6}\right)$.

$v(4) = \cos\left(\frac{2\pi}{3}\right) < 0$ and $a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) < 0$. Since the velocity and acceleration have the same signs at $t = 4$, speed of the particle is increasing at that time.

(d) $x(4) = x(0) + \int_0^4 \cos\left(\frac{\pi}{6}t\right)dt = -2 + \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\Big|_0^4 = -2 + \frac{6}{\pi}\left(\sin\left(\frac{4\pi}{6}\right) - \sin(0)\right)$.[1]

Notes:

1. No need to simplify further. If you insist, $= -2 + \frac{3\sqrt{3}}{\pi}$.

2012 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) $\left.\dfrac{dx}{dt}\right|_{t=2} = \dfrac{2}{e^2} > 0$, so the particle is moving to the right at $t = 2$. The slope of the path of the particle at $t = 2$ is $\dfrac{dy}{dx} = \left.\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{x=2} = \dfrac{\sin^2 2}{2/e^2} \approx 3.055$.

(b) $x(4) = x(2) + \int_2^4 \dfrac{dx}{dt}\,dt = 1 + \int_2^4 \dfrac{\sqrt{t+2}}{e^t}\,dt \approx 1.253$.

(c) Speed = $\left.\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,\right|_{t=4} \approx 0.575$.

$\vec{a}(4) = \left.\left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}\right)\right|_{t=4} \approx (-0.041, 0.989)$.

(d) Total distance traveled = $\int_2^4 \sqrt{\left(\dfrac{\sqrt{t+2}}{e^t}\right)^2 + \left(\sin^2(t)\right)^2}\,dt \approx 0.651$.

Question BC-3

See AB Question 3.

Question BC-4

(a) The tangent line is $y-15=8(x-1)$. At $x = 1.4$, $y=15+8(0.4)$ [1] $=18.2$, and $f(1.4)\approx 18.2$. [2]

(b) The midpoint sum is $0.2\cdot 10+0.2\cdot 13=4.6$. This approximates $f(1.4)-f(1)$, so $f(1.4)\approx 15+4.6$ [3] $=19.6$.

(c) $f(1.2)\approx 15+8\cdot 0.2=16.6$ and $f(1.4)\approx 16.6+12\cdot 0.2$ [4] $=19$.

(d) The second-degree Taylor polynomial is $T_2(x)=15+8\cdot(x-1)+\dfrac{20}{2}\cdot(x-1)^2$.

$T_2(1.4)=15+8\cdot(0.4)+10\cdot(0.4)^2$ [1] $=19.8$. Thus, $f(1.4)\approx 19.8$. [5]

Notes:

1. Can stop here.
2. It would be incorrect to write: $f(1.4)=18.2$, with an equal sign — such an error is usually penalized.
3. You could leave it as $f(1.4)\approx 15+(0.2\cdot 10+0.2\cdot 13)$. Either way, not $f(1.4)=...$ with an equal sign.
4. Can stop here; either way not $f(1.4)=...$
5. Again, do not use the equal sign here.

Question BC-5

See AB Question 5.

Question BC-6

(a) $\lim_{n\to\infty}\left|\dfrac{\frac{x^{2n+3}}{2n+5}}{\frac{x^{2n+1}}{2n+3}}\right| = \lim_{n\to\infty}\left|\dfrac{x^{2n+3}}{x^{2n+1}}\cdot\dfrac{2n+3}{2n+5}\right| = x^2 < 1 \quad \Leftrightarrow \quad -1 < x < 1$. Both at $x = -1$ and at $x = 1$ the series is an alternating series whose terms decrease in absolute value to 0, so both these series converge by the alternating series test. Therefore, the interval of convergence is $-1 \le x \le 1$.

(b) The alternating series error bound guarantees that the sum of the first two terms differs from $g\left(\frac{1}{2}\right)$ by the next non-zero term of the series, which is

$$\frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}.$$

(c) The series is $\frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} - \ldots + (-1)^n\frac{(2n+1)x^{2n}}{2n+3} + \ldots$.