# Be Prepared

for the

# AP

# Calculus Exam

Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

## Chapter 10. Annotated Solutions to Past Free-Response Questions



Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

# 2009 AB
## AP Calculus Free-Response
## Solutions and Notes

**Question AB-1**

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = 0.2 - 0.3 = -0.1$ miles/minute$^2$.[1]

(b) $\int_0^{12} |v(t)|\,dt$ is the total distance in miles Caren traveled on her trip to school from $t = 0$ to $t = 12$ minutes.

$\int_0^{12} |v(t)|\,dt = \frac{1}{2} \cdot 2 \cdot 0.2 + \frac{1}{2} \cdot 2 \cdot 0.2 + \frac{1}{2} \cdot 0.3 + 0.3 + 0.25 + 3 \cdot 0.2 + \frac{1}{2} \cdot 1 \cdot 0.2 =$
$1.8$ miles.[2]

(c) Caren turned around at $t = 2$ minutes. That is when her velocity changed from positive to negative.

(d) $\int_0^{12} w(t)\,dt = 1.6$ miles is the distance from Larry's home to school.

$\int_0^{12} v(t)\,dt = 1.4$ miles[3] is the distance from Caren's home to school. Caren lives closer to school.

**Notes:**

1. Or miles per minute per minute.

2. No need to calculate the final answer. Pay attention to the units on the vertical axis.

3. Start the calculation at $t = 5$, when Caren left home the second time, after getting her calculus homework.

**Question AB-2**

(a) $\int_0^2 R(t)\,dt$ [1] $= 980$ people.[2]

(b) $R'(t) = 2\cdot 1380t - 3\cdot 675\cdot t^2$. $R'(t) = 0$ at $t = 0$ and $t = 1.363$ hours.
$R(0) = 0$, $R(1.363) = 854.527$, and $R(2) = 120$. By the candidate test, $R(t)$ has the maximum at $t = 1.363$ hours.

(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2-t)R(t)\,dt = 387.5$ hours.

(d) Total wait time is $\int_0^2 w'(t)\,dt = \int_0^2 (2-t)R(t)\,dt = 760$. On average, a person waits $\frac{760}{980} = .776$ hours.

**Notes:**

1. Use the given function name in your formulas.
2 Or, if you are using symbolic antidifferentiation, $1380\cdot\frac{1}{3}\cdot 8 - 675\cdot\frac{1}{4}\cdot 16$.

**Question AB-3**

(a) It costs $\int_0^{25} 6\sqrt{x}dx = 500$ dollars to produce the cable. Mighty receives $25\,\text{meters} \cdot \dfrac{\$120}{\text{meter}} = \$3000$ for the cable. The profit is \$2500.

(b) $\int_{25}^{30} 6\sqrt{x}dx$ is the cost difference in dollars between manufactoring a 30 meter long cable and a 25 meter long cable or the additional cost to manufacture an additional 5 meters at the end of a 25 meter cable.[1]

(c) Profit is $P(k) = 120k - \int_0^k 6\sqrt{x}dx$ dollars.

(d) $P'(k) = 120 - 6\sqrt{k} \Rightarrow k = 400$. Since $P'(k) > 0$ for $0 < k < 400$ and $P'(k) < 0$ for $k > 400$, profit is a maximum when $k = 400$. The maximum profit is $P(400) = 120 \cdot 400 - \int_0^{400} 6\sqrt{x}dx = 16000$ dollars.

**Notes:**

1. No need to evaluate this integral.

---

**Question AB-4**

(a) Area= $\int_0^2 2x - x^2 dx = \left( x^2 - \dfrac{x^3}{3} \right)\Bigg|_0^2 = 4 - \dfrac{8}{3}$.

(b) Volume = $\int_0^2 \sin\left(\dfrac{\pi}{2}x\right) dx = -\dfrac{2}{\pi}\cos\left(\dfrac{\pi}{2}x\right)\Bigg|_0^2 = \dfrac{2}{\pi}(1+1) = \dfrac{4}{\pi}$.

(c) Volume = $\int_0^4 \left(\sqrt{y} - \dfrac{y}{2}\right)^2 dy$.[1]

**Notes:**

1. <u>Not</u> $\int_0^2 \left(2x - x^2\right)^2 dx$. The sections are perpendicular to the <u>*y*-axis</u>, not the *x*-axis.

---

**Question AB-5**

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5-3} = \dfrac{-2-4}{2} = -3$.

(b) $\int_2^{13} \left(3 - 5f'(x)\right) dx = \left(3x - 5f(x)\right)\Big|_2^{13} = (39-30) - (6-5) = 8$.[1]

(c) Left Riemann sum = $1 \cdot 1 + 4 \cdot 2 + (-2) \cdot 3 + 3 \cdot 5 = 18$.[1]

(d) The tangent line at x = 5 is $y = -2 + 3(x-5)$. On the tangent line, at $x$ = 7, $y$ = 4. Since $f''(x) < 0$ for all $x$ in [5, 8], the graph of $f$ is concave down on that interval, and the tangent line is above the curve. Thus, $f(7) \le 4$.

The slope of the secant line = $\dfrac{3-(-2)}{8-5} = \dfrac{5}{3}$. An equation of the secant line is $y = -2 + \dfrac{5}{3}(x-5)$. At $x$ = 7, $y = -2 + \dfrac{10}{3} = \dfrac{4}{3}$. The secant line is under the graph of $f$, so $f(7) \ge \dfrac{4}{3}$.

**Notes:**

1. Calculating the final answer is optional.

**Question AB-6**

(a) The graph of $f$ has points of inflection only at $x = -2$ and $x = 0$, since these are the only values of $x$ where $f'$ has local extrema.[1]

(b) $f(-4) = f(0) + \int_0^{-4} f'(x)\,dx = 5 - (8 - 2\pi) = 2\pi - 3.$

$f(4) = f(0) + \int_0^4 f'(x)\,dx =$

$5 + \int_0^4 \left(5e^{-x/3} - 3\right)dx = 5 + \left(-15e^{-x/3} - 3x\right)\Big|_0^4 = 5 + \left(-15e^{-4/3} - 12\right) - \left(-15\right) = 8 - 15e^{-4/3}.$

(c) Since $f'(x) \geq 0$ for $-4 < x < 3\ln\left(\frac{5}{3}\right)$, $f$ is increasing there. Since $f'(x) < 0$ for $3\ln\left(\frac{5}{3}\right) < x < 4$, $f$ is decreasing there. Therefore, $f$ has its absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.

**Notes:**

1. That is, $f'$ changes from decreasing to increasing and vice-versa at these values of $x$.

# 2009 BC
## AP Calculus Free-Response Solutions and Notes

**Question BC-1**

See AB Question 1.

---

**Question BC-2**

See AB Question 2.

---

**Question BC-3**

(a) $\frac{dy}{dt} = 0$ when $t$ = 0.367347. Let $B$ = 0.367347.[1] For $0 < t < B$, $\frac{dy}{dt} > 0$ and for $t > \text{B}$, $\frac{dy}{dt} < 0$. Therefore, $y$ is maximum at $t = B$.

$y(B) = y(0) + \int_0^B 3.6 - 9.8t \; dt \approx 12.061$ meters.

(b) $y(t) = y(0) + \int_0^t 3.6 - 9.8u \; du = 11.4 + 3.6t - 4.9t^2$.

$y(t) = 0$ at $t$ = 1.936256 seconds.[1] $A$ = 1.936 seconds.[1]

(c) Total distance traveled is $\int_0^A \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \approx 12.946$ meters.

(d) $\tan(\theta) = \left|\frac{dy}{dx}\right|_{t=A} = \left|\frac{dy}{dt} \Big/ \frac{dx}{dt}\right|_{t=A} = \left|\frac{3.6 - 9.8A}{0.8}\right| \approx 19.219131$.[1,2]

The path makes an angle of $\arctan(\tan\theta) \approx 1.519$ radians with the water's surface.[3]

**Notes:**

1. Store this value in your calculator.
2. We need the absolute value because $0 < \theta < \frac{\pi}{2}$. Here $\left.\frac{dy}{dx}\right|_{t=A} = \tan(\pi - \theta) = -\tan(\theta)$.
3. The work in this problem is easy to check. Plot the motion of the diver in parametric mode and verify your answers.

**Question BC-4**

(a) At the point $(-1, 2)$, $\frac{dy}{dx} = 6 - 2 = 4$. So, $y_{new} = 2 + 4 \cdot \frac{1}{2} = 4$. At $\left(-\frac{1}{2}, 4\right)$, $\frac{dy}{dx} = 6 \cdot \frac{1}{4} - \frac{1}{4} \cdot 4 = \frac{1}{2}$. So $y = 4 + \frac{1}{2} \cdot \frac{1}{2} = 4.25$.

(b) $T_2(x) = 2 + 4 \cdot (x+1) + \frac{-12}{2}(x+1)^2$.

(c) $\frac{dy}{dx} = x^2(6 - y)$. The constant solution $y = 6$ does not satisfy the initial condition. Separating the variables, $\int \frac{dy}{6-y} = \int x^2 dx$. Antidifferentiating, we have $-\ln|6-y| = \frac{x^3}{3} + C$. Substituting $x = -1$, $y = 2$ we get $-\ln|4| = -\frac{1}{3} + C \Rightarrow$ $C = \frac{1}{3} - \ln 4$. Therefore, $-\ln|6-y| = \frac{x^3}{3} + \frac{1}{3} - \ln 4 \Rightarrow \ln|6-y| = -\frac{x^3}{3} - \frac{1}{3} + \ln 4 \Rightarrow$ $6 - y = e^{-\frac{x^3}{3} - \frac{1}{3} + \ln 4} \Rightarrow y = 6 - e^{-\frac{x^3}{3} - \frac{1}{3} + \ln 4}$. 1

**Notes:**

1. Or: $-\ln|6-y| = \frac{x^3}{3} + C \Rightarrow 6 - y = Ae^{-\frac{x^3}{3}}$ (where $A = \pm e^{-C}$). Substituting $x = -1$, $y = 2$ we get $4 = Ae^{\frac{1}{3}} \Rightarrow A = 4e^{-\frac{1}{3}} \Rightarrow y = 6 - 4e^{-\frac{1}{3}}e^{-\frac{x^3}{3}}$.

---

**Question BC-5**

See AB Question 5.

**Question BC-6**

(a) $e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \ldots + \frac{(x-1)^{2n}}{n!} + \ldots$ 1

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{4!} \ldots + \frac{(x-1)^{2n-2}}{n!} + \ldots$ 2

(c) $\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{2n}}{(n+1)!}}{\frac{(x-1)^{2n-2}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^2}{n+1}\right| = 0 < 1$ for all *x*. The interval of convergence is all real numbers.

(d) $f'(x) = (x-1) + \frac{2}{3}(x-1)^3 + \ldots + \frac{(2n-2)(x-1)^{2n-3}}{n!} + \ldots$

$f''(x) = 1 + 2(x-1)^2 + \ldots + \frac{(2n-2)(2n-3)(x-1)^{2n-4}}{n!} + \ldots$

$f''$ is positive for all *x*, so the graph of $f$ has no points of inflection.

**Notes:**

1. Just substitute $(x-1)^2$ for *x* in the given series for $e^x$.

2. Note that $f(1) = 1$ is given; just divide each term by $(x-1)^2$.

# 2009 AB (Form B)
## AP Calculus Free-Response Solutions and Notes

**Question AB-1 (Form B)**

(a) $R(t) = 6 + \int_0^t \frac{1}{16}\left(3 + \sin\left(u^2\right)\right) du$

$R(3) = 6 + \int_0^3 \frac{1}{16}\left(3 + \sin\left(u^2\right)\right) du$ 🖩 $\approx 6.6108477$ [1] $\approx 6.611$ centimeters.

(b) $A(t) = \pi \cdot \left(R(t)\right)^2$

$\frac{dA}{dt} = 2\pi R(t) \frac{dR}{dt}$

$\left.\frac{dA}{dt}\right|_{t=3} = 2\pi R(3) \left.\frac{dR}{dt}\right|_{t=3} = 2\pi \cdot R(3) \cdot \left(\frac{1}{16}\left(3 + \sin 9\right)\right)$ 🖩 $\approx 8.858\ \text{cm}^2$ / year.

(c) $\int_0^3 A'(t)\, dt = A(3) - A(0) = \pi\left(\left(R(3)\right)^2 - \left(R(0)\right)^2\right)$ 🖩 $\approx 24.201\ \text{cm}^2$.

From $t = 0$ to $t = 3$ years, the area of a cross section of the tree at the given height changed by 24.201 $\text{cm}^2$.

**Notes:**

1. Save this value in your calculator.

**Question AB-2 (Form B)**

(a) $35+\int_0^5 f(t)\,dt$ 🖩 $\approx 26.495$ meters.

(b) At time $t = 4$ hours, the rate of change of the distance from the water's edge to the road was increasing at the rate of 1.007 meters/hour per hour.

(c) We want the minimum value of $f(t)$. $f'(t)=0$ at $t_1=0.6619$ and at $t_2=2.8404$. Using the candidate test:
$f(0)=-2,\ f(t_1)=-1.398,\ f(t_2)=-2.270,\ f(5)=-0.4803$. The minimum occurs at $t\approx 2.840$ hours.

(d) $35=26.495+\int_0^t g(p)\,dp$, where $t$ is the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

**Question AB-3 (Form B)**

(a) No. We can see from the graph that $\frac{f(x)-f(0)}{x-0}$ is equal or close to $\frac{2}{3}$ for $-4<x<0$. Therefore, $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}>0$. We can see from the graph that *f* is decreasing on (0, 3), so $\frac{f(x)-f(0)}{x-0}$ is negative for $0<x<3$.[1] Therefore, $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}\le 0$ (or doesn't exist). Thus $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist.

(b) Two. For the average rate of change of *f* on the interval [*a*, 6] to be 0, we need $\frac{f(6)-f(a)}{6-a}=0$, or $f(a)=1$. This happens twice: once between $-2$ and $-1$ and again between 0 and 2.

(c) Yes, *a* = 3. On the closed interval [3, 6], we have $\frac{f(6)-f(3)}{6-3}=\frac{1-0}{3}=\frac{1}{3}$. Since *f* is continuous on [3, 6] and differentiable on (3, 6), the Mean Value Theorem guarantees the existence of a number *c* such that $3<c<6$ and $f'(c)=\frac{1}{3}$.

(d) $g'(x)=f(x)$ and $g''(x)=f'(x)$. Since *f* is increasing on (–4, 0) and (3, 6), we have, $f'(x)>0$ on these intervals, so $g''(x)>0$ and the graph of *g* is concave up on $(-4, 0)$ and $(3, 6)$.[2]

**Notes:**

1. Is the graph alone a sufficient justification? We'll know for sure only when the rubric is published. It is probably safer to add the following:

   Indeed, $f''(x)>0$ on (0, 3), so $f'$ is increasing there; $f'(3)=0$, so $f'(x)<0$ on (0, 3), so *f* is decreasing there.

2. Or on [–4, 0] and [3, 6].

**Question AB-4 (Form B)**

(a) Area = $\int_0^4 \left(\sqrt{x}-\frac{x}{2}\right)dx = \left(\frac{2}{3}x^{3/2}-\frac{x^2}{4}\right)\Bigg|_0^4 = \frac{16}{3}-4$. [1]

(b) Volume = $\int_0^4 \left(\sqrt{x}-\frac{x}{2}\right)^2 dx = \int_0^4 x - x^{3/2} + \frac{x^2}{4}\,dx = \left(\frac{x^2}{2}-\frac{2}{5}x^{5/2}+\frac{x^3}{12}\right)\Bigg|_0^4 =$

$8-\frac{64}{5}+\frac{64}{12}$. [2]

(c) Volume = $\pi\int_0^4 \left(2-\frac{x}{2}\right)^2 - \left(2-\sqrt{x}\right)^2 dx$.

**Notes:**

1. $=\frac{4}{3}$.
2. $=\frac{8}{15}$.

**Question AB-5 (Form B)**

(a) $g'(x) = f'(x)e^{f(x)} \Rightarrow g'(1) = f'(1)e^{f(1)} = -4e^2$. $g(1) = e^2$. The tangent line equation is $y - e^2 = -4e^2(x-1)$.[1]

(b) Since $e^{f(x)} > 0$ for all $x$, the sign of $g'(x)$ is the same as the sign of $f'(x)$. Therefore, $g'(x)$ changes sign from positive to negative only at $x = -1$, so $g$ has a local maximum only at $x = -1$.

(c) $g''(-1) = e^{f(-1)}\left(\left(f'(-1)\right)^2 + f''(-1)\right)$. $e^{f(-1)} > 0$, $f'(-1) = 0$, and $f''(-1) < 0$ (because $f'(x)$ is decreasing around $x = -1$), so $g''(-1) < 0$,

(d) From Part (a), $g'(1) = -4e^2$. The average rate of change of $g'$ on [1, 3] is $\dfrac{g'(3) - g'(1)}{3-1} = \dfrac{0 + 4e^2}{2} = 2e^2$.

**Notes:**

1. Or $y = 5e^2 - 4e^2 x$.

**Question AB-6 (Form B)**

(a) $a(36) \approx \dfrac{v(40)-v(32)}{40-32} = \dfrac{7+4}{8} = \dfrac{11}{8}$ meters/second$^2$.

(b) $\int_{20}^{40} v(t)\,dt$ is the net change in meters of the position of the particle from $t = 20$ seconds to $t = 40$ seconds. The trapezoidal sum approximation is $\dfrac{-10+(-8)}{2}\cdot 5+\dfrac{-8+(-4)}{2}\cdot 7+\dfrac{-4+7}{2}\cdot 8$ meters.[1]

(c) Yes. The particle must change direction from right to left in the interval [8, 20] when its velocity changes from positive to negative. The particle must change direction from left to right in the interval [32, 40] when its velocity changes from negative to positive.

(d) The position of the particle at $t = 8$ $x(8) = x(0) + \int_0^8 v(t)\,dt = 7 + \int_0^8 v(t)\,dt$.

$a(t) = v'(t) > 0$ for $0 < t < 8$, so $v(t)$ is increasing and $v(t) > v(0) = 3$ on that time interval. Therefore, $\int_0^8 v(t)\,dt > 3\cdot 8 = 24$ and, $x(8) > 7 + 24 > 30$ meters.

**Notes:**

1. $= -75$ meters.

# 2009 BC (Form B)
## AP Calculus Free-Response Solutions and Notes

**Question BC-1 (Form B)**

(a) Area = $30\cdot 20-\int_0^{30} f\left(x\right)dx \approx 218.028$.[1]

(b) The volume of the cake $V=\frac{\pi}{2}\int_0^{30}\left(10\sin\left(\frac{\pi x}{30}\right)\right)^2 dx \approx 2356.194$ cm$^3$. There will be $0.05\cdot 2356.194\approx 117.810$ grams of chocolate in the cake.

(c) Perimeter = $30+\int_0^{30}\sqrt{1+\left(f'\left(x\right)\right)^2}\,dx \approx 81.804$ cm.

**Notes:**

1. Or $600-20\cdot\frac{30}{\pi}\cdot\left(-\cos\left(\frac{\pi x}{30}\right)\right)\Big|_0^{30}=600\left(1-\frac{2}{\pi}\right)$.

---

**Question BC-2 (Form B)**

See AB Question 2.

---

**Question BC-3 (Form B)**

See AB Question 3.

---

**Question BC-4 (Form B)**

(a) The graph of the curve goes through the origin when $1-2\cos\theta=0 \Rightarrow \theta=\frac{\pi}{3}$.

Area = $\frac{1}{2}\int_0^{\pi/3}(1-2\cos\theta)^2\,d\theta$.

(b) $x=(1-2\cos\theta)\cos\theta=\cos\theta-2\cos^2\theta \Rightarrow \frac{dx}{d\theta}=-\sin\theta+4\cos\theta\sin\theta$.

$y=(1-2\cos\theta)\sin\theta=\sin\theta-2\cos\theta\sin\theta \Rightarrow \frac{dy}{d\theta}=\cos\theta+2\sin^2\theta-2\cos^2\theta$. [1]

(c) At $\theta=\frac{\pi}{2}$, $r$ = 1, $x$ = 0, $y$ = 1, and $\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{2}{-1}=-2$. Tangent line equation is $y=1-2x$.

**Notes:**

1. Or $y=(1-2\cos\theta)\sin\theta=\sin\theta-\sin 2\theta \Rightarrow \frac{dy}{d\theta}=\cos\theta-2\cos 2\theta$.

---

**Question BC-5 (Form B)**

See AB Question 5.

---

**Question BC-6 (Form B)**

(a) This series is a geometric series with common ratio $x + 1$. It converges when $\left|x+1\right|<1$ that is, when $-2<x<0$.

(b) The sum = $\dfrac{1}{1-\left(x+1\right)}=-\dfrac{1}{x}$.

(c) $g\left(x\right)=\int_{-1}^{x}-\dfrac{1}{t}\,dt=-\ln\left|t\right|\Big|_{-1}^{x}=-\ln\left|x\right|$. Thus, $g\left(-\dfrac{1}{2}\right)=-\ln\left(\dfrac{1}{2}\right)$. [1]

(d) $h\left(x\right)=1+x^2+x^4+\ldots+x^{2n}+\ldots$

$$h\left(\frac{1}{2}\right)=f\left(\frac{1}{4}-1\right)=f\left(-\frac{3}{4}\right)=-\frac{1}{-\frac{3}{4}}=\frac{4}{3}.$$

**Notes:**

1. $=\ln 2$.