Be Prepared

for the

AP

Calculus Exam

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Chapter 10. Annotated Solutions to Past Free-Response Questions

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2008 AB
AP Calculus Free-Response
Solutions and Notes

Question AB-1

The graphs intersect at the points with *x*-coordinates of 0 and 2.

(a) Area of $R = \int_0^2 \sin(\pi x) - (x^3 - 4x)\, dx$ 🖩 ≈ 4.000. 📋1

(b) The line $y = -2$ intersects the graph of $y = x^3 - 4x$ at $x = 0.5392$ and again at $x = 1.6751$. Let $a = 0.5392$ and $b = 1.6751$. Area $= \int_a^b \left(-2 - \left(x^3 - 4x\right)\right) dx$. 📋2

(c) Volume = $\int_0^2 \left(\sin\left(\pi x\right) - \left(x^3 - 4x\right)\right)^2 dx$ 🖩 ≈ 9.978.

(d) Volume = $\int_0^2 \left(\sin\left(\pi x\right) - \left(x^3 - 4x\right)\right) \cdot \left(3 - x\right) dx$ 🖩 ≈ 8.370

📋 **Notes:**

1. Even though it is easy to antidifferentiate the integrand in this question, it is probably safer and faster to use your calculator to evaluate the integral.

2. No need to evaluate the integral.

Question AB-2

(a) Rate at $t = 5.5 \approx \dfrac{L(7)-L(4)}{7-4} = \dfrac{150-126}{3} = 8$ people per hour.

(b) Average value of $L(t)$ is $\dfrac{1}{4}\int_0^4 L(t)\,dt$. Using a trapezoidal sum, this is approximately $\dfrac{1}{4}\cdot\left(\left(\dfrac{120+156}{2}\right)+\left(\dfrac{156+176}{2}\right)\cdot 2+\left(\dfrac{176+126}{2}\right)\right)=155.25$ people.[1]

(c) $L'(t)$ must be 0 at least three times. L is continuous and so, by the Intermediate Value Theorem, there must be a time t_1 between $t = 0$ and $t = 1$, another time t_2 between $t = 3$ and $t = 4$, another time t_3 between $t = 4$ and $t = 7$, and another time t_4 between $t = 7$ and $t = 8$ when $L(t) = 145$. By the Mean Value Theorem,[2] there must be a time between t_1 and t_2, another time between t_2 and t_3, and a third time between t_3 and t_4 when $L'(t) = 0$.[3]

(d) $\int_0^3 r(t)\,dt \approx 972.784$. There were 973 tickets sold.

Notes:

1. It is OK not to calculate the sum.

2. Or Rolle's theorem.

3. Alternative justification: $L(1) > L(0)$ and $L(1) > L(4)$, so L has a local maximum on the interval (0, 4); $L(4) < L(3)$ and $L(4) < L(7)$, so L has a local minimum on the interval (3, 7); $L(7) > L(4)$ and $L(7) > L(8)$, so L has a local maximum on the interval (4, 8). $L'(t)$ must be 0 at these three local extrema.

Question AB-3

(a) We are given $\frac{dV}{dt} = 2000$. Since $V = \pi r^2 h$, $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$. At the instant in question, we have $r = 100$, $h = 0.5$, and $\frac{dr}{dt} = 2.5$. Substituting, $2000 = \pi 100^2 \frac{dh}{dt} + 2\pi 100 \cdot 0.5 \cdot 2.5$. Solving gives $\frac{dh}{dt} = \frac{2000 - 250\pi}{10000\pi}$ 🖩 ≈ 0.039 [1] centimeters per minute.

(b) Once the recovery device starts working, $\frac{dV}{dt} = 2000 - 400\sqrt{t}$. $\frac{dV}{dt} > 0$ for $t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$. Thus, the maximum volume occurs at $t = 25$ minutes after the device starts removing oil.

(c) $60000 + \int_0^{25} \left(2000 - 400\sqrt{t}\right) dt$

Notes:

1. The numeric answer is optional.

Question AB-4

(a) $x(t) = -2 + \int_0^t v(t)\,dt$. From the graph, $x'(t) < 0$ for $0 < t < 3$ and for $5 < t < 6$.

$x'(t) > 0$ for $3 < t < 5$. So a minimum could occur at $t = 3$ or at $t = 6$.

$x(3) = -2 - 8 = -10$ and $x(6) = -2 - 8 + 3 - 2 = -9$. The particle is at its leftmost position of -10 at $t = 3$.

(b) The position $x(t)$ is a continuous function.

t	0	3	5	6
$x(t)$	-2	-10	-7	-9
Reason	Given	From Part(a)	$-10+3=-7$	$-7-2=-9$

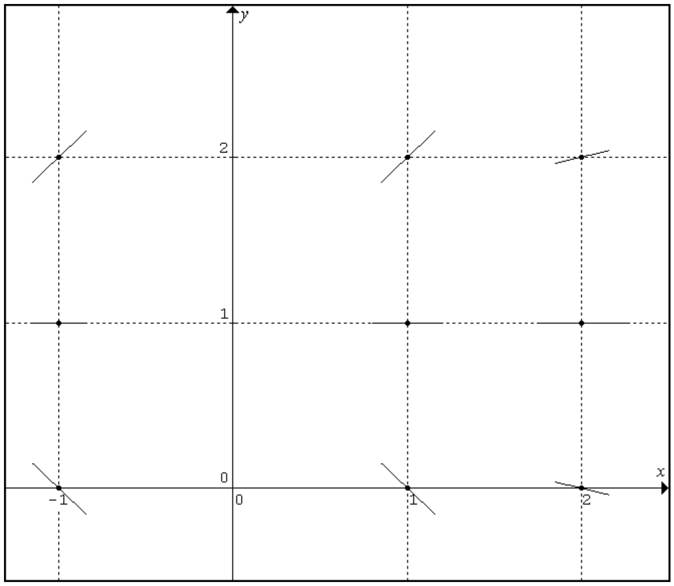
By the Intermediate Value Theorem, $x(t) = -8$ for some t, $0 < t < 3$, again for some t, $3 < t < 5$, and again for some t, $5 < t < 6$. So there are three times when $x(t) = -8$.

(c) On the interval $2 < t < 3$, $v(t) < 0$ and $v(t)$ is increasing. Therefore, the speed is decreasing.

(d) Since $a(t) = v'(t)$, the acceleration is negative when the velocity is decreasing. From the graph, this occurs for $0 < t < 1$ and $4 < t < 6$.

Question AB-5

(a)



(b) Separating variables, $\frac{dy}{y-1}=\frac{dx}{x^2}$, so $\int\frac{dy}{y-1}=\int\frac{dx}{x^2}$. Thus, $\ln|y-1|=-\frac{1}{x}+C$. From the initial condition, $\ln 1=-\frac{1}{2}+C \Rightarrow C=\frac{1}{2} \Rightarrow |y-1|=e^{\frac{1}{2}-\frac{1}{x}}$. Since the initial condition has $y<1$, $y-1=-e^{\frac{1}{2}-\frac{1}{x}}$ and $f(x)=1-e^{\frac{1}{2}-\frac{1}{x}}$. 1

(c) $\lim_{x\to\infty} f(x)=1-e^{\frac{1}{2}}=1-\sqrt{e}$.

Notes:

1. Or: $f(x)=1-\sqrt{e}e^{-\frac{1}{x}}$.

Question AB-6

(a) $f\left(e^2\right)=\dfrac{\ln\left(e^2\right)}{e^2}=\dfrac{2}{e^2}$ and $f'\left(e^2\right)=\dfrac{1-\ln\left(e^2\right)}{e^4}=-\dfrac{1}{e^4}$. An equation for the tangent line is $y-\dfrac{2}{e^2}=-\dfrac{1}{e^4}\left(x-e^2\right)$.

(b) $f'(x)=0$ at $x = e$. $f'(x)>0$ for $x < e$ and $f'(x)<0$ for $x > e$. Therefore, f has a local maximum at $x = e$.

(c) $f''(x)=\dfrac{x^2\cdot\left(-\dfrac{1}{x}\right)-(1-\ln x)(2x)}{x^4}=\dfrac{x(-1-2+2\ln x)}{x^4}=\dfrac{2\ln x-3}{x^3}$. Since the second derivative changes sign at $x=e^{\frac{3}{2}}$, the graph of f has a point of inflection there.

(d) The numerator of f approaches $-\infty$ and the denominator approaches 0, so $\lim\limits_{x\to 0^+} f(x)$ does not exist.[1]

Notes:

1. Or: $\lim\limits_{x\to 0^+} f(x)=-\infty$.

2008 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

See AB Question 2.

Question BC-3

(a) $P_1(x) = P(2) + P'(2)(x-2) = 80 + 128\cdot(x-2)$.
$h(1.9) \approx P_1(1.9) = 80 + 128\cdot(-0.1) = 67.2$. Since $h''(x) > 0$ for all x in the interval $1.9 \le x \le 2$, [1] this approximation is less than $h(1.9)$.

(b) $P_3(x) = 80 + 128\cdot(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$.
$h(1.9) \approx P_3(1.9) = 80 + 128\cdot(-0.1) + \frac{488}{6}(-0.1)^2 + \frac{448}{18}(-0.1)^3$. [2]

(c) Since the fourth derivative is increasing, its maximum on the interval $1.9 \le x \le 2$ occurs at x = 2. It is $\frac{584}{9}$. Therefore, the Lagrange error bound is

$$\left|\frac{\frac{584}{9}}{4!}(-.1)^4\right| \approx 2.7\cdot 10^{-4} < 3\cdot 10^{-4}.$$

Notes:

1. The graph of h is concave up.

2. ≈ 67.988.

Question BC-4

See AB Question 4.

Question BC-5

(a) Since $x = 3$ is a critical point and $f'(x) < 0$ for $x < 3$ and $f'(x) > 0$ for $x > 3$, f has a local minimum at $x = 3$.

(b) f is decreasing for $x < 3$ since $f'(x) < 0$ there. $f''(x) = (x-3)e^x + e^x = e^x(x-2)$. It is positive when $x > 2$. So f is decreasing and concave up for $2 < x < 3$.[1]

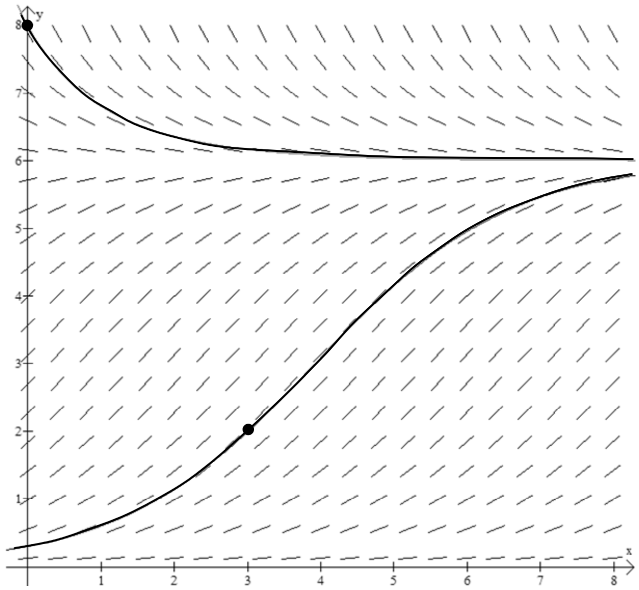
(c) $f(3) = 7 + \int_1^3 (x-3)e^x\,dx$. Using integration by parts with $u = x-3$ and $dv = e^x dx$, we have $du = dx$ and $v = e^x$. So,
$f(3) = f(1) + \int_1^3 (x-3)e^x\,dx = 7 + \left((x-3)e^x\right)\Big|_1^3 - \int_1^3 e^x\,dx =$
$7 + 0\cdot e^3 - (-2)e - (e^3 - e) = 7 + 3e - e^3$.

Notes:

1. Or $2 \le x \le 3$.

Question BC-6

(a)



(b) At (0, 8), $\frac{dy}{dt} = -2$. $y_{new} = 8 + (-2)\cdot\frac{1}{2} = 7$. At $\left(\frac{1}{2}, 7\right)$, $\frac{dy}{dt} = -\frac{7}{8}$.

$y_{new} = 7 + \left(-\frac{7}{8}\right)\cdot\frac{1}{2} = 7 - \frac{7}{16}$.

(c) $\frac{d^2y}{dt^2} = \frac{y}{8}\left(-\frac{dy}{dt}\right) + (6-y)\left(\frac{1}{8}\frac{dy}{dt}\right) = \frac{dy}{dt}\left(-\frac{y}{8} + \frac{3}{4} - \frac{y}{8}\right) = \frac{dy}{dt}\left(\frac{3}{4} - \frac{y}{4}\right)$. At $t = 0$,

$\frac{d^2y}{dt^2} = -2\cdot\left(\frac{3}{4} - 2\right) = \frac{5}{2}$. $P_2(t) = 8 - 2t + \frac{5}{4}t^2 \Rightarrow f(1) \approx P_2(1) = 8 - 2 + \frac{5}{4}$.[1]

(d) $6 < y \le 8$.

Notes:

1. = 7.25.

2008 AB (Form B)
AP Calculus Free-Response Solutions and Notes

Question AB-1 (Form B)

The graphs intersect at the points (0, 0) and (9, 3).

(a) Area $A_R = \int_0^9 \left(\sqrt{x} - \frac{x}{3} \right) dx = 4.5$.

(b) $x_1(y) = 3y;\ x2(y) = y^2$. Using washers,

Volume $= \pi \int_0^3 (3y+1)^2 - \left(y^2+1\right)^2 dy \approx 130.062$.

(c) Volume $= \int_0^3 \left(3y - y^2\right)^2 dy \approx 8.100$.

Question AB-2 (Form B)

(a) Distance = $\int_0^2 r(t)\,dt$ 🖩 ≈ 206.370 km.

(b) $\frac{d}{dt}g(x(t)) = \frac{d}{dx}g(x)\cdot\frac{dx}{dt}$. From Part (a), at t = 2 the car has traveled 206.37 km. $g'(206.37)$ 🖩 ≈ 0.05 liters/km.[1] At time t = 2, the speed of the car is $r(2) \approx 120.000$ km/hour. So, the rate of change of the number of liters of gasoline with respect to time is $0.05\ \frac{\text{liters}}{\text{km}}\cdot 120\ \frac{\text{km}}{\text{hour}} = 6$ liters/hour.

(c) Solving $r(t) = 80$ 🖩 gives $t = T = 0.33145$ hours.[2] At this time, the car has traveled $S = \int_0^T r(t)\,dt$ 🖩 ≈ 10.7940965[3] km. The amount of fuel consumed is $g(S) \approx 0.537$ liters.

Notes:

1. Or, if you use symbolic differentiation,
$$g'(x) = 0.05\left[\left(1 - e^{-\frac{x}{2}}\right) + x\cdot\frac{1}{2}e^{-\frac{x}{2}}\right] \Rightarrow g'(2) = 0.05.$$

2. Store this value as T in your calculator.

3. Store this value, too, as S and use your calculator to evaluate $g(S)$.

Question AB-3 (Form B)

(a) The area is approximately $8 \cdot 3.5 + 6 \cdot 7.5 + 8 \cdot 5 + 2 \cdot 1 = 115$ ft^2. Let A = 115.[1]

(b) Using the approximation from Part (a), volumetric flow F is approximated by $F = a \cdot v(t)$. The average value of F over [0, 120] is

$$\frac{1}{120}\int_0^{120} A \cdot v(t)\, dt \approx 1807.170 \text{ ft}^3\text{/min.}$$

(c) Area = $\int_0^{24} f(x)\,dx \approx 122.231$ ft^2. Let B = 122.231.[1]

(d) Average volumetric flow over [40, 60] is $\frac{1}{60-40}\int_{40}^{60} B \cdot v(t)\,dt \approx 2181.913$ ft^3/min. Since this is greater than 2100, the water must be diverted.

Notes:

1. Store in your calculator.

Question AB-4 (Form B)

(a) $f'(x) = 3\sqrt{4+9x^2}$ and $g'(x) = 3\cos(x)\sqrt{4+9\sin^2(x)}$.

(b) The slope is $g'(\pi) = 3\cos(\pi)\sqrt{4+9\sin^2(\pi)} = -6$. $g(\pi) = f(0) = 0$. Tangent line is $y = -6(x-\pi)$.

(c) $g'(x) = 0$ at $x = \frac{\pi}{2}$. Since $g'(x) > 0$ for $0 \le x < \frac{\pi}{2}$ and $g'(x) < 0$ for $\frac{\pi}{2} < x \le \pi$, g has a maximum at $x = \frac{\pi}{2}$.[1] $g\left(\frac{\pi}{2}\right) = f(1) = \int_0^3 \sqrt{4+t^2}\,dt$.

Notes:

1. You could also use the candidate test, evaluate $g(0) = 0$ and $g(\pi) = 0$, and note that $g\left(\frac{\pi}{2}\right) > 0$, so it's the maximum.

Question AB-5 (Form B)

(a) The graph of $y = g(x)$ has points of inflection at $x = 1$ and $x = 4$, since its derivative has local extrema at those points.

(b) From the graph of g', we see that g decreases from $x = -3$ to $x = -1$ and again from $x = 2$ to $x = 6$ since $g'(x) < 0$. From $x = -1$ to $x = 2$ and from $x = 6$ to $x = 7$, g is increasing since $g'(x) > 0$. Therefore, g has a local maximum at $x = 2$ where g' changes sign from positive to negative. g also has maxima at the endpoints. Evaluating, we get $g(-3) = 5 + \int_2^{-3} g'(t)\,dt = 5 - \frac{3}{2} + 4 = \frac{15}{2}$ and $g(7) = 5 + \int_2^{7} g'(t)\,dt = 5 - 4 + \frac{1}{2} = \frac{3}{2}$. Therefore, the absolute maximum value of g is $\frac{15}{2}$.

(c) $\dfrac{g(7) - g(-3)}{10} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$.

(d) $\dfrac{g'(7) - g'(-3)}{10} = \dfrac{1}{2}$. Since $g'(x)$ is not differentiable at $x = -1$ (nor at $x = 1$, nor at $x = 4$, but just one point is sufficient), the Mean Value Theorem does not apply to $g'(x)$ on [–3, 7].[1]

Notes:

1. Even though $g''(c) = \frac{1}{2}$ for any c, such that $-1 < c < 1$.

Question AB-6 (Form B)

(a) $2x+2+4y^3y'+4y'=0 \Rightarrow 2y^3y'+2y'=-x-1 \Rightarrow y'=\dfrac{dy}{dx}=\dfrac{-x-1}{2y^3+2}=\dfrac{-(x+1)}{2(y^3+1)}$.

(b) The slope at $(-2,1)$ is $\frac{1}{4}$. The tangent line is $y-1=\frac{1}{4}(x+2)$.

(c) For a vertical tangent, we need $y^3+1=0$ or $y = -1$, while $x \neq -1$. Substituting into the equation for the curve, $x^2+2x+1-4=5 \Rightarrow x^2+2x-8=0 \Rightarrow (x+4)(x-2)=0$. So the points are $(-4, -1)$ and $(2, -1)$.

(d) We would need $\frac{dy}{dx}=0$ where $y = 0$. So $\dfrac{-(x+1)}{2}=0$ and $x = -1$. Substituting $(-1, 0)$ into the equation of the curve, we get $1-2=5$, which is never true. So there is no point where the curve intersects the x-axis and has a horizontal tangent.

2008 BC (Form B)
AP Calculus Free-Response
Solutions and Notes

Question BC-1 (Form B)

(a) At $t = 4$ the acceleration vector is $\vec{a} = \left.\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right)\right|_{t=4} = (0.433, -11.872)$.

(b) $y(0) = 5 + \int_4^0 \frac{dy}{dt}\,dt$ 🖩 ≈ 1.601.

(c) Solving $3.5 = \sqrt{\left(\sqrt{3t}\right)^2 + \left(3\cos\left(\frac{t^2}{2}\right)\right)^2}$ 🖩 gives $t \approx 2.226$.

(d) Distance = $\int_0^4 \sqrt{\left(\sqrt{3t}\right)^2 + \left(3\cos\left(\frac{t^2}{2}\right)\right)^2}\,dt$ 🖩 ≈ 13.182.

Question BC-2 (Form B)

See AB Question 2.

Question BC-3 (Form B)

See AB Question 3.

Question BC-4 (Form B)

(a) $kx^2 - x^3 = 0 \Rightarrow x^2(k - x) = 0 \Rightarrow x = 0$ or $x = k$.

Area = $\int_0^k (kx^2 - x^3)\,dx = \left(\frac{kx^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^k = \frac{k^4}{3} - \frac{k^4}{4} = \frac{k^4}{12} = 2 \Rightarrow k = 24^{\frac{1}{4}}$.

(b) Volume = $\pi\int_0^k (kx^2 - x^3)^2\,dx$.

(c) Perimeter = $k + \int_0^k \sqrt{1 + (f'(x))^2}\,dx = k + \int_0^k \sqrt{1 + (2kx - 3x^2)^2}\,dx$.

Question BC-5 (Form B)

See AB Question 5.

Question BC-6 (Form B)

(a) Use the geometric series to generate the series for *f* :

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+...+x^n+...$$

Replace *x* by $-x^2$:

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-...+\left(-x^2\right)^n+...$$

Multiple by 2*x*:

$$\frac{2x}{1+x^2}=2x-2x^3+2x^5-2x^7+...+2\left(-1\right)^n x^{2n+1}+...$$

(b) At *x* = 1, the series is 2 – 2 + 2 – 2 + … Since $\lim_{n\to\infty}\left(2\left(-1\right)^n\right)$ does not exist, by the *n*-th term test, the series does not converge.

(c) Antidifferentiating the series for $\frac{2x}{1+x^2}$ we get the series for $\ln\left(1+x^2\right)$:

$$\ln\left(1+x^2\right)=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+...$$

(d) When $x=\frac{1}{2}$, $\frac{2x}{1+x^2}=\frac{5}{4}$. The series from Part (c) converges at $x=\frac{1}{2}$ because it is an alternating series with descreasing terms, and the magnitude of the terms approaches 0. we can use it to approximate $\ln\left(\frac{5}{4}\right)$: $\ln\left(\frac{5}{4}\right)=\frac{1}{4}-\frac{1}{32}+\frac{1}{192}-...$. By the alternating series error bound, $\left|\ln\left(\frac{5}{4}\right)-\left(\frac{1}{4}-\frac{1}{32}\right)\right|<\frac{1}{192}$. Therefore, $\left|A-\ln\left(\frac{5}{4}\right)\right|<\frac{1}{100}$, where $A=\frac{1}{4}-\frac{1}{32}$.