Be Prepared
for the

AP

Calculus Exam

Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

Chapter 10. Annotated Solutions to Past Free-Response Questions

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

2006 AB
AP Calculus Free-Response
Solutions and Notes

Question 1

Solving $\ln x = x - 2$, the graphs intersect at x = .158594 and x = 3.146193. Let a = .158594 and b = 3.146193.[1]

(a) $A = \int_a^b \ln x - (x-2)\,dx \approx 1.949$.[2]

(b) $V_{y=-3} = \pi \int_a^b (3 + \ln x)^2 - (3 + x - 2)^2\,dx \approx 34.199$.[2]

(c) By "washers": $V_{x=0} = \pi \int_{\ln a}^{\ln b} (y+2)^2 - (e^y)^2\,dy$;

by "shells": $V_{x=0} = 2\pi \int_a^b x(\ln x - (x-2))\,dx$.

Notes:

1. Store the intersection points in calculator variables and use those variables when calculating the integrals. See *Be Prepared*, page 256.

2. Use your calculator to evaluate the integrals; don't bother trying to antidifferentiate.

Question 2

(a) $\int_0^{18} L(t)dt \approx 1657.8237 \;\Rightarrow\;$ 1658 cars.

(b) Solving $L(t) = 150$ gives $t_1 \approx 12.42831$ and $t_2 \approx 16.121657$.[1] Average value of L on the closed interval $[t_1, t_2]$ is $\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} L(t)\,dt \approx 199.426$ cars per hour.[2]

(c) The number of cars turning left for $14 \le t \le 16$ hours is $\int_{14}^{16} L(t)\,dt \approx 412.266$ cars. $412.266 \cdot 500 = 206133 > 200000$. So, yes, a traffic light is needed.

Notes:

1. Store t_1 and t_2 in calculator variables and use those variables when calculating the integrals.

2. Be attentive to the units.

Question 3

(a) $g(4) = \int_0^4 f(t)dt = -1 + 4 = 3$; $g'(4) = f(4) = 0$; $g''(4) = f'(4) = \frac{-2-2}{5-3} = -2$.

(b) By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. From the graph of *f*, we see that $g'(x)$ changes sign from negative to positive at $x = 1$. Therefore, *g* has a relative minimum at $x = 1$.

(c) $g(10) = \int_0^{10} f(t)\,dt = \int_0^5 f(t)\,dt + \int_5^{10} f(t)\,dt = 2 + 2 = 4$.

$g(108) = 21\int_0^5 f(t)\,dt + \int_0^3 f(t)\,dt = 21\cdot 2 + 2 = 44$.

$g'(108) = f(108) = f(21\cdot 5 + 3) = f(3) = 2$.

The tangent line equation is $y - 44 = 2(x - 108)$.

Question 4

(a) Average acceleration = $\frac{49-5}{80-0}=\frac{11}{20}$ feet per second per second.[1]

(b) $\int_{10}^{70} v(t)\,dt$ is the net change in height of the rocket in feet from t = 10 seconds to t = 70 seconds. $\int_{10}^{70} v(t)\,dt \approx 22\cdot 20+35\cdot 20+44\cdot 20=2020$ feet.

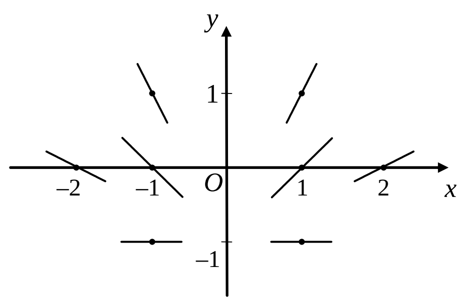
(c) For Rocket A, $v(80)=49$ ft/sec. For rocket B, $a(t)=v'(t)$ and $v(0)=2$ so $v(80)=2+\int_0^{80} a(t)\,dt = 2+\int_0^{80}\frac{3}{\sqrt{t+1}}dt = 2+6\sqrt{t+1}\Big|_0^{80} = 2+(54-6)=50$ ft/sec $\Rightarrow$ Rocket B is traveling faster.

Notes:

1. Or ft/sec^2.

Question 5

(a)



(b)

$\frac{dy}{dx}=\frac{1+y}{x} \Rightarrow \int\frac{dy}{1+y}=\int\frac{dx}{x} \Rightarrow \ln|1+y|=\ln|x|+C \Rightarrow y+1=\pm Ax$, where $A=e^C$. $y(-1)=1$ we have to choose the negative sign and $A=2 \Rightarrow$ $y+1=-2x \Rightarrow y=-2x-1$. Since $\frac{dy}{dx}$ has a discontinuity at x = 0, the domain is x < 0.

Question 6

(a) $g'(x) = ae^{ax} + f'(x) \;\Rightarrow\; g'(0) = a + f'(0) = a - 4$.
$g''(x) = a^2e^{ax} + f''(x) \;\Rightarrow\; g''(0) = a^2 + f''(0) = a^2 + 3$.

(b) $h(0) = \cos(0)\,f(0) = 2$. $h'(x) = -k\sin(kx)\,f(x) + \cos(kx)\,f'(x) \;\Rightarrow$
$h'(0) = -k\sin(0)\,f(0) + \cos(0)\,f'(0) = f'(0) = -4$. An equation for the tangent line to the graph of h at $x = 0$ is $y - 2 = -4x$.

2006 BC
AP Calculus Free-Response
Solutions and Notes

Question 1

See AB Question 1.

Question 2

See AB Question 2.

Question 3

(a) $\vec{a}(t)=\left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}\right) \Rightarrow \vec{a}(2)\approx(0.396, -0.741)$.[1]

Speed at $t = 2$ is $\left|\vec{v}(2)\right| \approx \left|(0.81734,\ 0.8889)\right| \approx 1.208$.

(b) If the tangent line is vertical, we must have

$\dfrac{dx}{dt}=0 \Rightarrow \arcsin\left(1-2e^{-t}\right)=0 \Rightarrow t=\ln 2$.[2]

(c)

$$m(t)=\frac{\dfrac{4t}{1+t^3}}{\arcsin\left(1-2e^{-t}\right)} \Rightarrow \lim_{t\to\infty} m(t)=0.$$

(d) $c=y(2)+\displaystyle\int_2^{\infty}\frac{dy}{dt}\,dt=-3+\int_2^{\infty}\frac{4t}{1+t^3}\,dt$.[3]

Notes:

1. No need to do symbolic differentiation — use your calculator.

2. Or use equation solver on your calculator: $t\approx 0.693$.

3. Since we are told a horizontal asymptote exists, it is sufficient to just write the answer. In general, if we were asked to show that a horizontal asymptote exists, we would need to show that $\lim_{t\to\infty} x(t)=\infty$ and $\lim_{t\to\infty} y(t)=c$ (or $\lim_{t\to T} x(t)=\infty$ and $\lim_{t\to T} y(t)=c$ for some T).

Question 4

See AB Question 4.

Question 5

(a) $\left.\frac{dy}{dx}\right|_{(-1,-4)} = 5 - \frac{6}{-6} = 6$.

$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + 6 \cdot \frac{1}{36} \cdot 6 = -9$.

(b) No. On the x-axis, $y = 0$, so $\frac{dy}{dx} = 5x^2 + 3 > 0$. But if $y = 0$ is a tangent, then we must have $\frac{dy}{dx} = 0$.

(c) $T_{2,f} = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$.

(d) At $x = -1$, $y = -4$, and slope = 6 $\Rightarrow x_1 = -\frac{1}{2}$, $y_1 = -4 + 6 \cdot \frac{1}{2} = -1$. Here, slope = $\frac{5}{4} - \frac{6}{-3} = \frac{13}{4}$. $y_2 = -1 + \frac{13}{4} \cdot \frac{1}{2} = \frac{5}{8}$. Thus, $f(0) \approx \frac{5}{8}$. [1]

Notes:

1. Or draw and fill a table:

	Point 0	Point 1	Point 2
x	–1	–0.5	0
y	–4	$-4 + 6 \cdot \frac{1}{2} = -1$	$-1 + \frac{13}{4} \cdot \frac{1}{2} = \frac{5}{8}$
$m = 5x^2 - \frac{6}{y-2}$	6	$\frac{5}{4} - \frac{6}{-3} = \frac{13}{4}$	

Question 6

(a) Ratio test: $\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{(n+1)x^{n+1}}{n+2}}{\dfrac{nx^n}{n+1}}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)x^{n+1}}{n+2}\cdot\dfrac{n+1}{nx^n}\right| = |x| < 1$. [1]

At $x=-1$ the series is $\sum\limits_{n=1}^{\infty}\dfrac{n}{n+1}$, which diverges because $\lim\limits_{n\to\infty}\dfrac{n}{n+1}\neq 0$. At $x=1$ the series diverges because $\lim\limits_{n\to\infty}\dfrac{(-1)^n n}{n+1}\neq 0$. [2] The interval of convergence is $-1 < x < 1$.

(b) $y'(0) = f'(0) - g'(0) = -\dfrac{1}{2} - \left(-\dfrac{1}{2}\right) = 0$. $y''(0) = f''(0) - g''(0) = \dfrac{4}{3} - \dfrac{1}{12} = \dfrac{15}{12}$.

y has a relative minimum at $x = 0$ since $y'(0) = 0$ and $y''(0) > 0$.

Notes:

1. Don't forget the limit and the absolute values.

2. The endpoints must be tested since we are asked for the interval of convergence, not just the radius.

2006 AB (Form B)
AP Calculus Free-Response
Solutions and Notes

Question 1

Solving $f(x) = 0 \Rightarrow x = -1.37312$. Let $a = -1.37312$.[1]

(a) $A_R = \int_a^0 f(x)\,dx \approx 2.903$.[2]

(b) $V_{y=-2} = \pi \int_a^0 \left(2 + f(x)\right)^2 - 2^2\,dx = 59.361$.[2]

(c) $f(0) = 3$ and $f'(0) = -\frac{1}{2} \Rightarrow$ an equation for the tangent line at $x = 0$ is

$y = -\frac{1}{2}x + 3$. The graph of f intersects this tangent line at $x = 0$, and again at

$x = 3.38987$. $A_S = \int_0^b \left(-\frac{1}{2}x + 3 - f(x)\right)dx$, where $b = 3.38987$.

Notes:

1. Store the intersection points in calculator variables and use those variables when calculating the integrals. See *Be Prepared*, page 256.

2. Use your calculator to evaluate the integrals; don't bother trying to antidifferentiate.

Question 2

(a) The graph of f is concave down on the interval $1.7 < x < 1.9$ since f' is decreasing on that interval.

(b) By the Fundamental Theorem of Calculus, $f(x) = 5 + \int_0^x f'(t)\,dt$. f can have an absolute maximum either at one of the endpoints or where $f'(x)$ changes from positive to negative, at $x \approx 1.77245$. $f(0) = 5$; $f(3) \approx 5.579$; $f(1.77245) \approx 5.679$. f has an absolute maximum at $x = 1.772$.

(c) $f(2) = 5.623$ and $f'(2) \approx -.459$. An equation for the tangent line at $x = 2$ is $y = 5.623 - .459(x-2)$.

Question 3

(a) $f'(x) = 2ax$. $f(4) = 16a = 1$ and $f'(4) = 8a = 1 \Rightarrow 16a = 8a$, which is impossible for $a \neq 0$.

(b) $g'(x) = 3cx^2 - \frac{x}{8}$. $g(4) = 64c - 1 = 1 \Rightarrow c = \frac{1}{32}$. Substituting c in g' gives $g'(4) = \frac{3}{32} \cdot 4^2 - \frac{4}{8} = 1$.

(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{x}{8}\left(\frac{3}{4}x - 1\right)$. $g'(x) < 0$ for $0 < x < \frac{4}{3} \Rightarrow g$ is not increasing between $x = 0$ and $x = 4$.

(d) $h(4) = \frac{4^n}{k} = 1 \Rightarrow k = 4^n$. $h'(x) = \frac{nx^{n-1}}{k} \Rightarrow h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = 1 \Rightarrow n = 4$. $k = 4^4 = 256$. Thus $h(x) = \frac{x^4}{256}$ and $h'(x) = \frac{4x^3}{256} = \frac{x^3}{64}$. Property (i): $h(0) = 0$ and $h'(0) = 0$. Property (iii): Since $h'(x) > 0$ for $x > 0$, h is increasing between $x = 0$ and $x = 4$.

Question 4

(a) $f'(22) = \dfrac{3-15}{24-20} = -3$ calories per minute per minute.[1]

(b) For $4 < t < 24$, the greatest rate of increase of f is $\dfrac{15-9}{16-12} = \dfrac{3}{2}$. For $0 < t < 4$, $f'(t) = -\dfrac{3}{4}t^2 + 3t$ and $f''(t) = -\dfrac{3}{2}t + 3$. $f''(2) = 0$ and f'' changes sign from positive to negative there, so f' has a maximum at $t = 2$. $f'(2) = 3 > \dfrac{3}{2} \Rightarrow$ f is increasing at its greatest rate at $t = 2$.

(c) Total number of calories burned from $t = 6$ to $t = 18$ is
$\int_6^{18} f(t)\,dt = \int_6^{12} f(t)\,dt + \int_{12}^{16} f(t)\,dt + \int_{16}^{18} f(t)\,dt = 9\cdot 6 + \dfrac{15+9}{2}\cdot 4 + 15\cdot 2 =$[2]
$54 + 48 + 30 = 132$.

(d) The existing setting burns an average of $\dfrac{132}{12} = 11$ calories per minute. To increase that to 15 calories per minute requires $c = 4$.

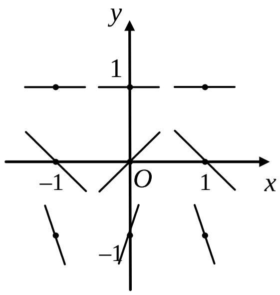
Notes:

1. Or calories/minute^2.

2. You could leave the answer in this form for Part (c) — no need to simplify — but you need that number for Part (d).

Question 5

(a)



(b) $c = 1$.

(c) $(y-1)^{-2}\dfrac{dy}{dx} = \cos(\pi x) \Rightarrow \int (y-1)^{-2}\,dy = \int \cos(\pi x)\,dx \Rightarrow$

$\dfrac{(y-1)^{-1}}{-1} = \dfrac{1}{\pi}\sin(\pi x) + C$. $y(1) = 0 \Rightarrow \dfrac{(0-1)^{-1}}{-1} = \dfrac{1}{\pi}\sin(\pi) + C \Rightarrow C = 1$. Thus,

$\dfrac{(y-1)^{-1}}{-1} = \dfrac{1}{\pi}\sin(\pi x) + 1 \Rightarrow y - 1 = -\dfrac{1}{\frac{1}{\pi}\sin(\pi x)+1} \Rightarrow f(x) = 1 - \dfrac{1}{\frac{1}{\pi}\sin(\pi x)+1}$.

Question 6

(a) $\int_{30}^{60} |v(t)|\,dt$ is the total distance in feet traveled by the car from t = 30 seconds to t = 60 seconds. The trapezoidal approximation to this integral is $\frac{14+10}{2}\cdot 5+\frac{10+0}{2}\cdot 15+\frac{0+10}{2}\cdot 10$ feet[1] $=60+75+50=185$ feet.

(b) $\int_0^{30} a(t)\,dt=\int_0^{30} v'(t)\,dt=v(30)-v(0)=-14-(-20)=6$ ft/sec. This is the net change in velocity in feet per second of the car from t = 0 seconds to t = 30 seconds.

(c) Yes. v is continuous with $v(35)=-10$ and $v(50)=0$. By the Intermediate Value Theorem, there must be a time t between 35 and 50 when $v(t)=-5$.

(d) Yes. v is differentiable (its derivative is $a(t)$), and $\frac{v(25)-v(0)}{25-0}=0$. By the Mean Value Theorem, there must be a time between t = 0 and t = 25 when $v'(t)=a(t)=0$.

Notes:

1. You can leave the answer in this form — no need to simplify.

2006 BC (Form B)
AP Calculus Free-Response
Solutions and Notes

Question 1

See AB Question 1.

Question 2

(a) At $t = 1$, $\dfrac{dy}{dx} = \dfrac{\sec\left(e^{-1}\right)}{\tan\left(e^{-1}\right)} \approx 2.781$. The tangent line equation is $y = -3 + 2.781(x-2)$.

(b) $\vec{a}\left(t\right) = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}\right) \Rightarrow$ $\vec{a}\left(1\right) \approx \left(-0.423, -0.152\right)$.[1]

Speed at $t = 1$ is $\left|\vec{v}(t)\right| = \sqrt{\sec^2\left(e^{-1}\right) + \tan^2\left(e^{-1}\right)}$ ≈ 1.139.

(c) Total distance traveled = $\int_1^2 \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ ≈ 1.059.[2]

(d) No. $x(t) = 2 + \int_1^t \tan\left(e^{-t}\right) dt \Rightarrow$ $x(0) \approx 1.224$. Since $0 < e^{-t} < 1$ for $t > 0$,

$\dfrac{dx}{dt} = \tan\left(e^{-t}\right) > 0$ for $t > 0 \Rightarrow x(t)$ is increasing for $t > 0$

$\Rightarrow x(t) > x(0) \approx 1.224 > 0$ for $t > 0$.

Notes:

1. No need to do symbolic differentiation — use your calculator.

2. Use your calculator to evaluate the integral; don't bother trying to antidifferentiate.

Question 3

See AB Question 3.

Question 4

See AB Question 4.

Question 5

(a) $\frac{1}{y}\frac{dy}{dx} = (6-2x) \Rightarrow \int \frac{dy}{y} = \int (6-2x)\,dx \Rightarrow \ln|y| = 6x - x^2 + C$. $y(4) = 1 \Rightarrow$ $0 = 24 - 16 + C \Rightarrow C = -8 \Rightarrow |y| = e^{6x-x^2-8}$. Since $y > 0$ at the initial point, $f(x) = e^{6x-x^2-8}$

(b) $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x\to\infty} g'(x) = 0$.[1]

(c) The graph of g has a point of inflection at $y = \frac{3}{2}$, because $\frac{dy}{dx}$, as a function of y, is a parabola, which reaches its maximum halfway between its zeroes, $y = 0$ and $y = 3$. $\left.\frac{dy}{dx}\right|_{y=\frac{3}{2}} = 2\cdot\frac{3}{2}\left(3-\frac{3}{2}\right)$[2] $= \frac{9}{2}$.

Notes:

1. $\lim_{x\to\infty} g(x)$ is the carrying capacity, and $\lim_{x\to\infty} g'(x)$ is 0 for any logistic curve.

2. You can stop here.

Question 6

(a) $-3x^2+6x^5-9x^8+...+(-1)^n 3n\cdot x^{3n-1}+...$

(b) The given series is the Maclaurin series for $f'(x)$ at $x=\frac{1}{2}$. $f'(x)=\frac{-3x^2}{\left(1+x^3\right)^2}$. The series converges to $f'\left(\frac{1}{2}\right)=\frac{-\frac{3}{4}}{\left(\frac{9}{8}\right)^2}=-\frac{16}{27}$.

(c) $x-\frac{x^4}{4}+\frac{x^7}{7}-\frac{x^{10}}{10}+...+(-1)^n\frac{x^{3n+1}}{3n+1}+...$

(d) $\int_0^{1/2} f(t)\,dt \approx \frac{1}{2}-\frac{1}{4\cdot 16}+\frac{1}{7\cdot 128}$ [1] $=\frac{435}{896}$. When $x=\frac{1}{2}$, the series in Part (c) is alternating with decreasing absolute values of terms, and the *n*-th term approaches 0 as $n\to\infty$. By the alternating series error bound, the magnitude of the error does not exceed the absolute value of the first omitted term, which is $\frac{1}{10\cdot 2^{10}}=\frac{1}{10\cdot 1024}<\frac{1}{10000}$.

Notes:

1. You can leave it like this — no need to calculate.